Calculus II, Summer 2006
Midterm 2

Instructions
1. This take home midterm is due Thursday, July 27th at 8 PM. Bring it to class or drop it off at my office before then. NO LATE SUBMISSIONS WILL BE ACCEPTED.
2. Write all solutions in the space provided, and use the back pages if you have to.
3. For full credit you must show all work, including all substitutions and integration by parts that are performed. Providing only an answer will result in very few marks.
4. You may use calculators for doing numerical computations and graphing only. If you have a fancy calculator you can't use it for doing integrals or derivatives.
5. If you don't understand the wording of any of the questions or what exactly they're asking just ask me, but I won't give help on how to answer the questions.
6. Since it's open book you can discuss the questions amongst fellow classmates (please don't consult outside help), but everyone must write up their own solutions. NEVER COPY STRAIGHT FROM ANOTHER PERSON'S SOLUTIONS.

1. (20 pts.) Describe the monotonicity and boundedness of each of the following sequences. For boundedness I want to know if it's bounded above, bounded below, or both, and I want you to give actual bounds.
   
   (a) \( a_n = \frac{n}{n^2 + n - 1} \)
   
   (b) \( b_n = (-1)^n \frac{n^2}{1 + n^2} \)
   
   (c) \( c_n = \frac{\sin n}{\sqrt{n}} \)
   
   (d) \( d_n = \frac{n!}{(n+2)!} \)

   a) \( a_n \geq 0 \) since both numerator and denominator are \( \geq 0 \)
      \[ a_n = \frac{\sqrt{n}}{1 + \sqrt{n} - \frac{n}{2}} \leq \frac{\sqrt{n}}{1 + \sqrt{n}} \leq 1 \]
      for all \( n \), and also \( a_n \) decreases to 0.

   b) \( a_n \) is oscillating because of the \((-1)^n\) term.
      It's not bounded above or below since \( \frac{n^3}{1 + n^2} \to \infty \).

   c) \( |\sin n| \leq 1 \Rightarrow |c_n| \leq \frac{1}{\sqrt{n}} \Rightarrow 1 < c_n < 1 \)
      \[ \Rightarrow -1 < c_n < 1 \text{ so it's bounded.} \]
      However it still oscillates because \( \sin(n) \) switches from \(+\) to \(-\).

   d) \( d_n = \frac{n!}{(n+2)!} = \frac{n}{(n+1)(n+2)} \times \frac{1}{n!} \Rightarrow 0 \leq d_n \leq \frac{1}{6} \)
      \& \( d_n \) decreases to zero.
2. (20 pts.) If $0 < r < 1$, then it's pretty clear that \( \lim_{n \to \infty} r^n = 0 \). This means that as \( n \) gets big the sequence \( r^n \) gets extremely close to zero, and as we discussed in class extremely close means as close as we want. Let \( \epsilon > 0 \) be a really small number. Show me that by choosing a number \( N \) sufficiently large, then for all \( n \geq N \) you have \( r^n < \epsilon \), or in other words \( r^n \) eventually gets within and \( \epsilon \)-band of zero and stays there. This is the precise version of what \( \lim_{n \to \infty} r^n = 0 \) really means.

Given \( \epsilon > 0 \) want \( r^n < \epsilon \)

\[ \Rightarrow \log(r^n) < \log \epsilon \]

\[ \Rightarrow n \log r < \log \epsilon \]

\[ \Rightarrow n > \frac{\log \epsilon}{\log r} \]

Hence choose \( N > \frac{\log \epsilon}{\log r} \), and then for all \( n \geq N \)

you're guaranteed that \( r^n < \epsilon \).
3. (40 pts.) Determine whether the following sequences converge or diverge. If they converge, find their limit.

(a) 
\[ a_n = \frac{n \cos n}{n^2 + 1} \]

(b) 
\[ b_n = (\sqrt{n+1} - \sqrt{n})\sqrt{n + 1/2} \]

(c) 
\[ c_n = \frac{4n^3 - 3n^2 + n - 7}{5n^3 + 2n - 6} \]

(d) 
\[ d_n = \frac{2^{2n-1}}{7n+2} \]

(a) 
\[-1 \leq \cos n \leq 1 \implies \frac{-n}{n^2 + 1} \leq \frac{n \cos n}{n^2 + 1} \leq \frac{n}{n^2 + 1} \]
\[ \to 0 \text{ as } n \to \infty \]

So by the Pinching Theorem \( a_n \) converges and
\[ \lim_{n \to \infty} a_n = 0. \]

(b) 
\[ b_n = \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \]
\[ = \sqrt{n+1} (\sqrt{n+1}^2 - \sqrt{n}^2) \]
\[ = \sqrt{n+1} + \sqrt{n} \]
\[ = \frac{\sqrt{n+1}}{\sqrt{n+1} + \sqrt{n}} + \frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \]
\[ = \frac{\sqrt{n+1} \sqrt{n}}{\sqrt{n+1} \sqrt{n} + \sqrt{n} \sqrt{n}} \to \frac{\sqrt{1}}{\sqrt{1} + \sqrt{1}} = \frac{1}{2} \text{ as } n \to \infty. \]
(c) \[ C_n = \frac{4n^3 - 3n^2 + n + 7}{5n^3 + 2n + 6} = \frac{1}{n^3} \left( \frac{4n^3 - 3n^2 + n + 7}{5n^3 + 2n + 6} \right) \]

\[ = 4 - \frac{3}{n} + \frac{1}{n^2} - \frac{7}{n^3} \]

\[ \Rightarrow \frac{4 - 0 + 0 + 0}{5 + 2 + 6/n^3} = \frac{4}{5} \text{ as } n \to \infty. \]

So, the sequence converges and has limit \( \frac{4}{5} \).

(d) \[ d_n = \frac{2^{3n} - 1}{7^{n+2}} = \frac{2^{-1}}{7^2} \cdot \frac{2^{3n}}{7^n} \]

\[ = \frac{1}{2 \cdot 7^2} \cdot \left( \frac{2}{7} \right)^n \]

\[ = \frac{1}{2 \cdot 7^2} \cdot \left( \frac{8}{7} \right)^n \]

\[ \Rightarrow \infty \text{ as } n \to \infty \text{ since } \left( \frac{8}{7} \right)^n \to \infty. \]

Thus \( d_n \) diverges.
4. (50 pts.) Compute the following limits:

(a) \[ \lim_{x \to 0} \frac{\sin x - x}{x^3} \]

(b) \[ \lim_{x \to \infty} xe^{-x^2} \]

(c) \[ \lim_{x \to 0^+} \frac{\ln x}{\sqrt{x}} \]

(d) \[ \lim_{x \to \infty} \left( \frac{x}{x + 1} \right)^x \]

(e) \[ \lim_{x \to \infty} \frac{x^3}{x^2 - 1} - \frac{x^3}{x^2 + 1} \]

(a) \[ \lim_{x \to 0} \frac{\sin x - x}{x^3} = \frac{0}{0} \quad \text{so use L'Hopital} \]

\[ = \lim_{x \to 0} \frac{\cos x - 1}{3x^2} \]

\[ = \lim_{x \to 0} \frac{-\sin x}{6x} = \frac{-1}{6} \]

(b) \[ \lim_{x \to \infty} xe^{-x^2} = \infty \cdot 0 \quad \text{limit} \]

\[ \lim_{x \to \infty} x^3 e^{-x^2} = \lim_{x \to \infty} \frac{x^3}{e^{x^2}} = \frac{\infty}{\infty} \quad \text{limit} \]

\[ = \lim_{x \to \infty} \frac{3x^2}{2xe^{x^2}} = \lim_{x \to \infty} \frac{3}{2e^{x^2}} = \frac{0}{0} \quad \text{limit} \]

\[ = \lim_{x \to \infty} \frac{6x}{2e^{x^2} + 4x^2 e^{-x^2}} = \lim_{x \to \infty} \frac{6}{2e^{x^2} + 4x^2} = \frac{6}{\infty} = 0 \]

\[ = \frac{6}{\infty} = 0. \]
(e) \[ \lim_{x \to 0^+} \frac{\ln x}{\sqrt{x}} = -\infty \]
\[ \lim_{x \to 0^+} \ln x = -\infty, \]
\[ \lim_{x \to 0^+} \sqrt{x} = 0 \]

(d) \[ \lim_{x \to \infty} \left( \frac{x}{x+1} \right)^x = \infty \]
\[ \lim_{x \to \infty} \left( \frac{x}{x+1} \right)^x = \lim_{x \to \infty} e^{\ln \left( \frac{x}{x+1} \right)^x} \]
\[ = \lim_{x \to \infty} e^{x \ln \left( \frac{x}{x+1} \right)} \]
\[ = e^{\lim_{x \to \infty} x \ln \left( \frac{x}{x+1} \right)} \]
\[ = e^{0} = 1 \]

\[ \lim_{x \to \infty} x \ln \left( \frac{x}{x+1} \right) \to 0 \quad \text{limit} \]
\[ \Rightarrow \ln(1) = 0 \]

\[ x \ln \left( \frac{x}{x+1} \right) = \ln(x) - \ln(x+1) = \frac{\ln \left( \frac{x}{x+1} \right)}{x} = \frac{\ln x - \ln(x+1)}{x} \]

\[ \Rightarrow \lim_{x \to \infty} \frac{\ln x - \ln(x+1)}{x} = \lim_{x \to \infty} \frac{\frac{1}{x} - \frac{1}{x+1}}{\frac{x^2}{x^2}} \]
\[ = \lim_{x \to \infty} \left( \frac{1}{x} - \frac{1}{x+1} \right) \]
\[ = \lim_{x \to \infty} -x \frac{1}{x(x+1)} = \lim_{x \to \infty} \frac{x+1 - x}{x(x+1)} \]
\[ = \lim_{x \to \infty} -\frac{x}{x+1} = -1 \]

\[ \Rightarrow \lim_{x \to \infty} \left( \frac{x}{x+1} \right)^x = e^{-1} \]
\[
\lim_{{x \to \infty}} \frac{x^3}{{x^2} - 1} - \frac{x^3}{{x^2} + 1} = \infty - \infty \quad \text{limit.}
\]

\[
= \lim_{{x \to \infty}} \frac{x^3({x^2} + 1) - x^3({x^2} - 1)}{({x^2} + 1)({x^2} - 1)}
\]

\[
= \lim_{{x \to \infty}} \frac{2x^3}{{x^4} - 1}
\]

\[
= 0
\]
5. (20 pts.) Suppose we have a rod of infinite length, which we can think of as being the real line. Let \( a > 0 \), and suppose we initially spread 1 unit of heat evenly across the interval \([-a, a]\). After time \( t \) the heat will spread out across the whole rod, and in fact physics gives us the following formula for the amount of heat at point \( x \) at time \( t \):

\[
T(x, t) = \frac{1}{a\sqrt{\pi t}} \int_0^a e^{-\frac{(x-u)^2}{4t}} du
\]

Compute for me \( \lim_{a \to 0} T(x, t) \). This represents the amount of heat at point \( x \) at time \( t \) when all the heat is initially concentrated at the origin.

As \( a \to 0 \), \( \int_0^a e^{-\frac{(x-u)^2}{4t}} du \to 0 \) since you're integrating over nothing.

But also the \( a \) on the bottom goes to 0.

Hence

\[
\lim_{a \to 0} \frac{1}{a\sqrt{\pi t}} \int_0^a e^{-\frac{(x-u)^2}{4t}} du
\]

\[
= \frac{1}{\sqrt{\pi t}} \lim_{a \to 0} \frac{\int_0^a e^{-\frac{(x-u)^2}{4t}} du}{a} = \frac{0}{0} \text{ limit}
\]

(L'Hopital)

\[
= \frac{1}{\sqrt{\pi t}} \lim_{a \to 0} \frac{\frac{\partial}{\partial a} \left( \int_0^a e^{-\frac{(x-u)^2}{4t}} du \right)}{\frac{\partial}{\partial a} (a)}
\]

(Fundamental Theorem of Calculus)

\[
= \frac{1}{\sqrt{\pi t}} \lim_{a \to 0} \frac{e^{-\frac{(x-a)^2}{4t}}}{a} = e^{-\frac{x^2}{4t}}
\]
6. (30 pts.) Determine whether the following integrals converge or diverge. For those that are convergent compute the actual value:

(a) \[ \int_{-\infty}^{\infty} \frac{x}{1+x^2} \, dx \]

(b) \[ \int_{0}^{\infty} \frac{dx}{x^2 + 6} \]

(c) \[ \int_{1}^{\infty} \frac{\ln x}{x^2} \, dx \]

(a) \[ \int_{-\infty}^{\infty} \frac{x}{1+x^2} \, dx = \int_{-\infty}^{0} \frac{x}{1+x^2} \, dx + \int_{0}^{\infty} \frac{x}{1+x^2} \, dx \]

\[ \int_{0}^{\infty} \frac{x}{1+x^2} \, dx = \frac{1}{2} \int_{0}^{\infty} \frac{2x}{1+x^2} \, dx \]

\[ = \frac{1}{2} \left[ \ln(1+x^2) \right]_0^\infty \]

\[ = \frac{1}{2} \left( \ln(\infty) - \ln(1) \right) \]

\[ = \infty \]

Thus one part of the integral diverges, so the whole thing does.

(b) \[ \int_{0}^{\infty} \frac{dx}{x^2 + 9} = \left. \frac{1}{3} \arctan \left( \frac{x}{3} \right) \right|_0^\infty \]

\[ = \left. \frac{1}{3} \left( \arctan(\infty) - \arctan(0) \right) \right|_0^\infty \]

\[ = \frac{1}{3} \left( \pi/2 - 0 \right) = \frac{\pi}{6} \]

(c) \[ \int_{1}^{\infty} \frac{\ln x}{x^3} \, dx = \int_{1}^{\infty} \frac{\ln x}{x^3} \, \frac{dx}{x} \]

\[ = \int_{0}^{\infty} \frac{u}{e^{2u}} \, du \]

\[ u = \ln x \Rightarrow x = e^u \]

\[ du = \frac{dx}{x} \]
\[ = \int_0^\infty u e^{-2u} \, du \]
\[ = -\frac{u}{2} e^{-2u} \bigg|_0^\infty + \frac{1}{2} \int_0^\infty e^{-2u} \, du \]  \hspace{1cm} \text{(Integration by Parts)}
\[ = -\left( \lim_{x \to \infty} \frac{x}{2} e^{-2x} - 0 e^0 \right) + \frac{-1}{4} e^{-2u} \bigg|_0^\infty \]
\[ = -\lim_{x \to \infty} \frac{x}{2} \frac{1}{e^{2x}} - \frac{1}{4} (e^{-\infty} - e^0) \]
\[ = -\lim_{x \to \infty} \frac{1}{2} \frac{1}{e^{2x}} + \frac{1}{4} \]
\[ = 0 + \frac{1}{4} = \frac{1}{4} \]
7. (30 pts.) Use the comparison theorem to determine if each integral is convergent or divergent.

(a) \[ \int_0^1 \frac{e^{-x}}{x^{1/3}} \, dx \]

(b) \[ \int_1^\infty \frac{dx}{x + e^{2x}} \]

(c) \[ \int_1^\infty \frac{\sqrt{1 + \sqrt{x}}}{\sqrt{x}} \, dx \]

(a) For \( 0 < x < 1 \), \( e^x > 1 \) \( \Rightarrow \) \( e^{-x} = \frac{1}{e^x} < 1 \)

\( \Rightarrow \frac{e^{-x}}{x^{1/3}} < \frac{1}{x^{1/3}} \)

\( \Rightarrow 0 < \int_0^1 \frac{e^{-x}}{x^{1/3}} \, dx \leq \int_0^1 \frac{1}{x^{1/3}} \, dx \)

Converges by the p-test

\( \Rightarrow \int_0^1 \frac{e^{-x}}{x^{1/3}} \, dx \) converges by comparison theorem.

(b) For \( x > 1 \), \( x + e^{2x} > e^{2x} \) \( \Rightarrow \frac{1}{x + e^{2x}} < \frac{1}{e^{2x}} \)

\( \Rightarrow 0 < \int_1^\infty \frac{dx}{x + e^{2x}} < \int_1^\infty \frac{dx}{e^{2x}} \)

\( S_a = \int_a^\infty \frac{dx}{x + e^{2x}} \) Converges.

\( \int_1^\infty \frac{e^{-2x}}{x} \, dx = \int_1^\infty e^{-2x} \, dx \)

\( = -\frac{1}{2} e^{-2x} \bigg|_1^\infty \)

\( = -\frac{1}{2} (e^{-\infty} - e^{-2}) = \frac{1}{2} e^2 \)

(c) \( u = \sqrt{x} \) \( \Rightarrow \) \( du = \frac{dx}{2\sqrt{x}} \), \( x = 1 \) \( \Rightarrow u = 1 \)

\( x = \infty \) \( \Rightarrow u = \infty \)

[Diverges]

\( \int_1^\infty \frac{\sqrt{1 + \sqrt{x}}}{\sqrt{x}} \, dx = \int_1^\infty 2 \sqrt{1 + u} \, du \geq \int_1^\infty 2 \, du = \infty. \)
8. (30 pts.) Determine whether or not the follow series are convergent or divergent. If they're convergent sum them up.

(a) \[ \sum_{k=0}^{\infty} \frac{1 - 4^k}{5^k} \]

(b) \[ \sum_{n=1}^{\infty} \left[ \sin \left( \frac{1}{n} \right) - \sin \left( \frac{1}{n+1} \right) \right] \]

(c) \[ \sum_{n=1}^{\infty} \frac{1}{5 + 2^{-n}} \]

(a) \[
\sum_{k=0}^{\infty} \frac{1 - 4^k}{5^k} = \sum_{k=0}^{\infty} \frac{1}{5^k} - \frac{4^k}{5^k} = \sum_{k=0}^{\infty} \left( \frac{1}{5} \right)^k - \sum_{k=0}^{\infty} \left( \frac{4}{5} \right)^k
\]
\[
= \frac{1}{1 - \frac{1}{5}} - \frac{1}{1 - \frac{4}{5}} = \frac{5}{4} - 5 = -\frac{15}{4}
\]

(b) It's a telescoping series.
\[
s_1 = \sin \left( \frac{1}{1} \right) - \sin \left( \frac{1}{2} \right)
\]
\[
s_2 = \sin \left( \frac{1}{1} \right) - \sin \left( \frac{1}{2} \right) + \sin \left( \frac{1}{2} \right) - \sin \left( \frac{1}{3} \right) = \sin \left( \frac{1}{1} \right) - \sin \left( \frac{1}{3} \right)
\]
\[
s_3 = \sin \left( \frac{1}{1} \right) - \sin \left( \frac{1}{2} \right) + \sin \left( \frac{1}{2} \right) - \sin \left( \frac{1}{3} \right) + \sin \left( \frac{1}{3} \right) - \sin \left( \frac{1}{4} \right) = \sin \left( \frac{1}{1} \right) - \sin \left( \frac{1}{4} \right)
\]
\[
\vdots
\]
\[
s_n = \sin \left( \frac{1}{1} \right) - \sin \left( \frac{1}{n+1} \right)
\]
\[
\Rightarrow \lim_{n \to \infty} s_n = \sin \left( \frac{1}{1} \right) - \lim_{n \to \infty} \sin \left( \frac{1}{n+1} \right)
\]
\[
= \sin \left( \frac{1}{1} \right) - \sin \left( \frac{1}{n+1} \right) = \sin \left( \frac{1}{1} \right)
\]
(c) \[ \sum_{n=1}^{8} \frac{1}{5+2^{-n}} \]

the sequence \[ \frac{1}{5+2^{-n}} \rightarrow \frac{1}{5} \]

so the terms of the series don't go to zero as \( n \to \infty \), hence the series diverges.
9. (30 pts.) Determine which of the following sequences converge or diverge. DON’T try to sum them up if they converge.

(a) \[ \sum_{n=1}^{\infty} \frac{\arctan n}{n^4} \]

(b) \[ \sum_{n=1}^{\infty} \frac{\sin^2 n}{n^{1/4}} \]

(c) \[ \sum_{k=1}^{\infty} \frac{1}{k \ln k} \]

(a) For \( n \geq 1 \), \( 0 \leq \arctan(n) \leq \frac{\pi}{2} \)

\[ \Rightarrow 0 \leq \frac{\arctan(n)}{n^4} \leq \frac{\pi}{2n^4} \]

\[ \Rightarrow 0 \leq \sum_{n=1}^{\infty} \frac{\arctan(n)}{n^4} \leq \sum_{n=1}^{\infty} \frac{\pi}{2n^4} < \infty \text{ by p-test} \]

Hence \( \sum_{n=1}^{\infty} \frac{\arctan(n)}{n^4} \) converges.

(b) \( 0 \leq \sin^2 n \leq 1 \)

\[ \Rightarrow 0 \leq \frac{\sin^2 n}{n^{1/4}} \leq \frac{1}{n^{3/4}} = \frac{1}{n^{3/2}} \]

\[ \Rightarrow 0 \leq \sum_{n=1}^{\infty} \frac{\sin^2 n}{n^{1/4}} \leq \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} < \infty \text{ } \Rightarrow \text{ by p-test} \]

Hence \( \sum_{n=1}^{\infty} \frac{\sin^2 n}{n^{1/4}} < \infty \).

(c) The terms a_k = \frac{1}{k \ln k} are decreasing but positive, so the integral test applies.

\[ \int_{1}^{\infty} \frac{dx}{x \ln x} = \int_{1}^{\infty} \frac{1}{\ln x} \frac{dx}{x} \]  
\[ u = \ln x \quad du = \frac{1}{x} \, dx \]

\[ = \int_{0}^{\infty} \frac{1}{u} \, du \]

which diverges by p-test \( \Rightarrow \sum_{k=1}^{\infty} \frac{1}{k \ln k} \) diverges.
10. (30 pts.) Determine which of the following sequences converge or diverge. DON'T try to sum them up if they converge.

(a) \[ \sum_{n=1}^{\infty} \frac{n^n}{5^{2n+3}} \]

(b) \[ \sum_{n=1}^{\infty} e^{-n} n! \]

(c) \[ \sum_{n=1}^{\infty} \frac{(n+2)!}{n!10^n} \]

(a) Root test. \[ a_n = \frac{n^n}{5^{2n+3}} \Rightarrow (a_n)^{\frac{1}{n}} = \frac{(n^n)^{\frac{1}{n}}}{(5^{2n+3})^{\frac{1}{n}}} = \frac{n}{5^{\frac{2n+3}{n}}} \]

\[ \lim_{n \to \infty} (a_n)^{\frac{1}{n}} = \lim_{n \to \infty} \frac{n}{5^{2+3\frac{1}{n}}} = \frac{\infty}{5^{2+0}} = \infty > 1 \]

\Rightarrow the series diverges by the root test.

(b) \[ \sum_{n=1}^{\infty} e^{-n} n! \]

Ratio Test. \[ a_n = e^{-n} n! \]

\[ \Rightarrow a_{n+1} = e^{-n-1} (n+1)! \]

\[ \Rightarrow \frac{a_{n+1}}{a_n} = \frac{e^{-n-1} (n+1)!}{e^{-n} n!} = e^{-1} \frac{n+1}{n} \]

\[ \Rightarrow \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = e^{-1} < 1 \]

\Rightarrow the series converges by ratio test.
(c) **Ratio Test**

\[ a_n = \frac{(n+2)!}{n! \cdot 10^n} = \frac{(n+2)(n+1)}{10^n} \]

\[ \Rightarrow a_{n+1} = \frac{(n+3)(n+2)}{10^{n+1}} \]

\[ \Rightarrow \frac{a_{n+1}}{a_n} = \frac{(n+3)(n+2)}{10^{n+1}} \div \frac{(n+2)(n+1)}{10^n} \]

\[ = \frac{(n+3)(n+2)}{10^{n+1}} \times \frac{10^n}{(n+2)(n+1)} \]

\[ = \frac{n+3}{n+1} \cdot \frac{1}{10} \]

\[ \Rightarrow \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{1}{10} \cdot \frac{n+3}{n+1} = \frac{1}{10} < 1 \]

\[ \Rightarrow \text{the series converges by the ratio test.} \]
11. (30 pts.) The Fibonacci sequence is defined as follows: let $a_1 = 1$, $a_2 = 1$ and for $n \geq 3$ define $a_n = a_{n-1} + a_{n-2}$.

When a sequence is defined like this, with the next value depending on the previous values, it’s called a recursive sequence.

(a) Write down the numerical values of $a_n$ for $3 \leq n \leq 10$.
(b) What’s the monotonicity of the sequence?
(c) Is it bounded above? Bounded below?
(d) Does $a_n$ converge?
(e) Create a new sequence $b_n$ defined by $b_n = a_{n+1}/a_n$. Write down the numerical value of $b_n$ for $1 \leq n \leq 10$.
(f) Show that $b_{n+1} = 1 + 1/b_{n-1}$.

(g) Assume that $b_n$ converges to a limit $L$. Use the equation from (f) to find out what $L$ is. (Hint: Since $b_n$ converges to $L$, when $n$ is big $b_{n-1}$ and $b_{n-1}$ are pretty much $L$. Use that and the equation to find what $L$ is.)

\[ a_3 = 2, \quad a_4 = 3, \quad a_5 = 5, \quad a_6 = 8, \quad a_7 = 13, \quad a_8 = 21, \quad a_9 = 34, \quad a_{10} = 55 \]

\[ \text{b) Increasing.} \]

\[ \text{c) Bounded below but not bounded above.} \]

\[ \text{d) No, it's increasing without bound.} \]

\[ b_1 = \frac{a_2}{a_1} = \frac{1}{1} = 1 \]

\[ b_2 = \frac{a_3}{a_2} = \frac{2}{1} = 2 \]

\[ b_3 = \frac{3}{2} = 1.5 \]

\[ b_4 = \frac{5}{3} = 1.666 \]

\[ b_5 = \frac{8}{5} = 1.6 \]

\[ b_6 = \frac{13}{8} = 1.625 \]

\[ b_7 = \frac{21}{13} = 1.615 \]

\[ b_8 = \frac{34}{21} = 1.619 \]

\[ b_9 = \frac{55}{34} = 1.618 \]

\[ b_{10} = \frac{89}{55} = 1.618 \]
(f) \[ b_n = \frac{a_n}{a_{n-1}} = \frac{a_n + a_{n-2}}{a_{n-1}} \]
\[ = 1 + \frac{a_{n-2}}{a_{n-1}} \]
\[ = 1 + \frac{1}{b_{n-2}} \]

(g) \[ \lim_{n \to \infty} b_n = L, \quad \lim_{n \to \infty} \frac{1}{b_{n-2}} = L \]
\[ \Rightarrow \lim_{n \to \infty} b_n = 1 + \lim_{n \to \infty} \frac{1}{b_{n-2}} \]
\[ \Rightarrow L = 1 + \frac{1}{L} \]
\[ \Rightarrow L^2 = L + 1 \]
\[ \Rightarrow L^2 - L - 1 = 0 \]
\[ \Rightarrow L = \frac{1 \pm \sqrt{1 - 4(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2} \]

From part (e), we see that \( \frac{1 + \sqrt{5}}{2} \) is the right answer.

\( \left( \frac{1 - \sqrt{5}}{2} < 0, \text{ which can't be right since it's a positive sequence} \right) \).

Note \( \frac{1 + \sqrt{5}}{2} = 1.618033 \).
12. (20 pts.) The Fibonacci sequence again. Let $a_n$ be the same sequence as in the last question.

(a) Show that $\frac{1}{a_{n-1}a_n} = \frac{1}{a_{n-1}a_n} - \frac{1}{a_na_{n+1}}$.

(b) Use part (a) to show that

$$\sum_{n=2}^{\infty} \frac{1}{a_{n-1}a_{n+1}} = 1$$

(a) \[
\frac{1}{a_{n-1}a_n} - \frac{1}{a_na_{n+1}} = \frac{a_{n}a_{n+1} - a_{n}a_{n-1}}{a_{n}^2a_{n-1}a_{n+1}} = \frac{a_{n}(a_{n+1} - a_{n-1})}{a_{n}^2a_{n-1}a_{n+1}} = \frac{a_{n}a_{n}}{a_{n}^2a_{n-1}a_{n+1}} = \frac{1}{a_{n-1}a_{n+1}}
\]

(b) Part (a) says that

$$\sum_{n=1}^{\infty} \frac{1}{a_{n-1}a_{n+1}} = \sum_{n=1}^{\infty} \left( \frac{1}{a_{n-1}a_{n}} - \frac{1}{a_{n}a_{n+1}} \right)$$

which is a telescoping series.

\[
S_1 = \frac{1}{a_0a_1} = \frac{1}{a_0a_1} - \frac{1}{a_1a_2}
\]

\[
S_2 = \left( \frac{1}{a_0a_1} - \frac{1}{a_1a_2} \right) + \left( \frac{1}{a_1a_2} - \frac{1}{a_2a_3} \right) = \frac{1}{a_0a_1} - \frac{1}{a_2a_3}
\]

\[
S_3 = S_2 + \frac{1}{a_2a_3} - \frac{1}{a_3a_4} = \frac{1}{a_0a_1} - \frac{1}{a_3a_4}
\]

$$\vdots$$

\[
S_n = \frac{1}{a_{n-1}a_n} - \frac{1}{a_na_{n+1}}
\]

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{1}{a_0a_1} - \frac{1}{a_na_{n+1}} = \frac{1}{a_0a_1} - \frac{1}{\infty} = \frac{1}{a_0a_1} = 1$$