1. (a) What is the purpose of the Sturm sequence algorithm?

(b) Explain why we can compute the determinant of a symmetric $n \times n$ tridiagonal matrix and the determinants of all its principal minors using only on the order $n$ arithmetic operations. If you write down a formula, explain its origin.

(c) Suppose you wish to find the largest eigenvalue of a symmetric tridiagonal matrix. Explain how you can find an upper bound using Gerschgorin’s theorem.
2. Consider a $m \times n$ matrix $A$.

   (a) Explain how to find a Householder matrix $Q$ such that $QA$ has zero elements in the first column except for its first element.

   (b) What is the form of $Q$ and how do we know it has orthonormal columns?

   (c) Is it possible to find more than one such $Q$?

   (d) Explain how to find a Householder matrix $Q$ such that the 11 element of $QA$ is the same as that of $A$ and such that all elements of the first column of $QA$ except for the first two equal 0.

   (e) Computing the product of an $m \times m$ matrix times a $m \times n$ matrix generally requires $m^2n$ multiplications. Explain how we can multiply a Householder matrix $Q$ by a $m \times n$ matrix $A$ using much fewer multiplications.
3. (a) Given a $m \times n$, $m > n$, matrix $A$ and a $m$-vector $b$, explain what the corresponding linear least squares problem is. Under what condition will this problem have a unique solution?

(b) Explain two different methods to solve such a problem and explain which one is preferred and why.

(c) How do we most conveniently solve such a problem in MATLAB?
4. (a) Suppose we wish to compute the Lagrange interpolation polynomial of a given function \( f(x) \). Assume that we work on the interval \([-1, +1]\). Why is it better to use the Chebyshev points instead of equidistant points if the number of points is large?

(b) What are the Chebyshev polynomials \( T_k(x) \)?

(c) How do we know that \( T_k(x), \ k = 0, 1, \ldots, n \) form a basis for \( \mathcal{P}_n \), the space of all polynomials of degree \( n \) or less.

(d) Write \( 1+x+x^2 \) as a linear combination of Chebyshev polynomials.
5. Let $A$ be an arbitrary symmetric tridiagonal matrix.

(a) Show that $B := A^2 + I$ always is symmetric, positive definite.

(b) What can we say about the sparsity of $B$?

(c) Consider the linear system $Bx = b$. How much storage is needed to solve this system with Cholesky’s algorithm?

(d) Estimate the number of arithmetic operations as a function of $n$, the order of the matrix $A$. 