1. Polynomial interpolation creates polynomials which matches the values of a given function $f(x)$ and possibly some of its derivatives at a set of $k + 1$ distinct points $x_0, \ldots, x_k$.

   (a) What is Lagrange interpolation and, in particular, what is the degree of the resulting polynomial that matches $f(x_i)$ as a function of $k$?

   (b) How can we show that such a polynomial exists and that it is unique?

   (c) What is Hermite interpolation? What is the degree of the Hermite interpolation polynomial for the set $x_0, \ldots, x_k$? Explain your answer.

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   (a) Lagrange interpolation constructs, given a function $f(x)$ and set of $k + 1$ distinct points, a polynomial $p_k(x)$ of degree $k$ such that $p(x_i) = f(x_i), i = 0, \ldots, k$. It can be constructed quite explicitly by using the $L_i(x)$ polynomials given in the text book.

   (b) Existence follows from the formula mentioned above. Now suppose that there is a second interpolation polynomial $q_k(x)$ which also matches the values of $f(x)$ at all these points. Then $p_k(x) - q_k(x)$, which is a polynomial of degree $k$, must vanish at $k + 1$ points and this difference must therefore be identically zero.

   There is an alternative argument. The interpolation problem can be written as a linear system of algebraic equations with the coefficients of the polynomial being the $k + 1$ unknowns. We know that the solution exists by the construction above for any values of the $f(x_i)$. We can then conclude that the matrix of this problem must be invertible and it follows from linear algebra that the solution is unique.

   (c) Hermite interpolation involves finding a polynomial of degree $2k + 1$ which matches the value of $f(x)$ at the $k + 1$ points and with a
derivative which matches the derivative of \( f(x) \) at all these points. Note that a polynomial of degree \( 2k + 1 \) has \( 2k + 2 \) coefficients and that we have that same number of conditions. An explicit formula for the Hermite interpolation polynomial is given in the text book; this settles the question of existence. Uniqueness can be handled just as in the Lagrange case.

2. An \( n \times n \) matrix \( H := I - 2vv^T \) is known as a Householder transformation. Here \( v \) is a non-zero column vector with \( n \) components.

(a) Explain why the Euclidean norm of \( Ha \) is the same as that of any vector \( a \), a column vector of length \( n \).

(b) Given arbitrary \( n \)-vectors \( a \) and \( b \), with the same Euclidean norm, explain how to find the vector \( v \) such that \( Ha = b \).

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(a) By a direct computation we see that \( H^T = H \) and that \( H^2 = I \).

\[
\|Ha\|_2^2 = a^T H^T H a = a^T a = \|a\|_2^2.
\]

(b) See page 151 of the text book. By the basic geometry related to the Householder transformation, we find that \( v \) can be chosen as \( a - b \).

YOUR NAME:

3. The maximum norm of a vector \( x \) is defined by \( \|x\|_\infty := \max_{1 \leq i \leq n} |x_i| \).

(a) Show that this norm satisfies the triangle inequality.

(b) How is the corresponding matrix norm \( \|A\|_\infty \) defined? Here \( A \) is an \( n \times n \) matrix.

(c) Is it true that \( \|A + B\|_\infty \leq \|A\|_\infty + \|B\|_\infty \)? If so, prove this fact.

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(a) \[ \max_i |x_i + y_i| \leq \max_i (|x_i| + |y_i|) \leq \max_i |x_i| + \max_j |x_j|. \]

(b) \[ \|A\|_\infty := \max_{\|x\|_\infty = 1} \|Ax\|_\infty. \]

(c) Yes. \[ \|A + B\|_\infty = \max_{\|x\|_\infty = 1} \|Ax + Bx\|_\infty \leq \max_{\|x\|_\infty = 1} \|Ax\|_\infty + \max_{\|y\|_\infty = 1} \|By\|_\infty. \]
4. Consider Newton’s method to solve a non-linear equation \( f(x) = 0 \).

(a) Does Newton’s method always converge from any initial guess \( x_0 \)?

(b) Assume that the iteration converges. Does Newton’s method then provide an upper and lower bound for the root to which it converges?

(c) Newton’s method often is rapidly convergent. Give an example where this is not the case. What is a condition on \( f(x) \) in such a case?

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(a) It is pretty easy to construct \( f(x) \) and \( x_0 \) so that Newton’s method diverges or alternates between two different values forever.

(b) Newton’s algorithm often approaches a root from one side and then there is no interval given by the iterates which is guaranteed to contain the root. Note that the bisection and the Illinois methods do provide a sequence of intervals, with shrinking length, which contains the root. Newton’s method does not and that complicates the development of a good stopping criterion.

(c) Slow convergence happens exactly if the first derivative of the function \( f(x) \) also vanishes at the root. An example is provided by \( f(x) = x^2 \).

YOUR NAME:

5. Let \( A \) be a \( n \times n \) matrix with real elements. \( A \) is strictly diagonally dominant if

\[
|a_{ii}| > \sum_{j \neq i} |a_{ij}|, \quad i = 1, \ldots, n.
\]

(a) Show, by using one of Gerschgorin’s theorems, that such a matrix always is invertible.

(b) When using Gaussian elimination and the matrix is strictly diagonally dominant, the algorithm can be simplified. Explain how.

(c) Assume that the matrix is diagonally dominant and tridiagonal but not necessarily symmetric. It is then known that it can be factored using only \( C_1 n + C_2 \) multiplications. Give an accurate
estimate of the constant $C_1$. Also give an accurate estimate of the number of divisions and additions/subtractions.

(a) The issue is if the matrix can have a zero eigenvalue. This is ruled out since the origin of the complex plane does not lie in any of the Gerschgorin circles.

(b) Pivoting is not needed for a strictly diagonally dominant matrix.

(c) Given that the matrix is tridiagonal and no pivoting is needed, we see that the first step of the factorization only involves the four elements in the upper left corner. We have to do the following operations: $\ell := a_{21}/a_{11}$, $a_{22}^{new} := a_{22} - a_{12} \star \ell$. In the next step we again have to deal with only four elements namely $a_{22}^{new}$, $a_{23}$, $a_{32}$, and $a_{33}$. Again only one division, one multiplication, and one subtraction is required. Since there are $n - 1$ steps in the entire factorization, $C_1 = 1$. 