1. Consider the approximation problem defined by
\[
\min_{p_2} \| f - p_2 \|_2, \quad \text{where} \quad p_2 \in P_2
\]
and where \( \| f \|_2 := (\int_1^{-1} |f(x)|^2dx)^{1/2} \). Here \( P_2 \) is the set of polynomials of degree 2. Find \( p_2(x) \) for \( f(x) = x^3 \).

2. Find \( p_2 \) for \( f(x) = e^x + x^2 \). In this case, you will face integrals that you cannot compute exactly. Approximate these integral by using a good adaptive quadrature rule from the MATLAB library.

3. Change the norm to
\[
(\int_1^{-1} |f(x)|^2dx)^{1/2} / (\int_1^{-1} \sqrt{1-x^2} dx)^{1/2}
\]
Solve the approximation problem as in the first problem with respect to this norm for \( f(x) = x^3 \).

4. Now consider the norm \( \| f \|_\infty := \max_{-1 \leq x \leq 1} |f(x)| \). Solve this approximation problem for \( P_2 \) and \( f(x) = x^3 \).

5. Solve \( \min_{p_2} \| x^3 - x + 2 - p_2 \|_\infty \).

6. Solve \( \min_{p_2} \| x^4 - p_2 \|_\infty \).

7. Solve \( \min_{p_2} \| x^3 - p_2 \|_\infty \) but where the norm is redefined as \( \max_{0 \leq x \leq 1} |f(x)| \).

8. Show that the following formula defines polynomials which are multiples of the Legendre polynomials:
\[
(d/dx)^j (x^2 - 1)^j.
\]
9. Consider a best approximation problem of the following kind:

\[
\min_{s_1} \| f(x) - s_1 \|_2^2,
\]

where \( s_1 \) is a linear spline defined on a set of knots \(-1 = x_0 < x_1 < \ldots < x_{n-1} < x_n = 1\). Set up a linear system of equations to determine the values of the minimizing linear spline at the \( x_i \). Can this system be solved quickly and if so explain why.