

YOUR NAME:

Sample Questions from Earlier Exams.

1. (a) What is a Householder matrix? What does such a matrix “look like”?
 - (b) Why are such matrices also called Householder reflectors?
 - (c) How can we use Householder matrices to transform a real, symmetric matrix into a symmetric, tridiagonal matrix with the same eigenvalues as the given symmetric matrix? What is the main idea that makes this algorithm work? How do we know that the eigenvalues of the matrices are indeed the same.
 - (d) Suppose we try to use the same algorithm for a nonsymmetric matrix. Can we then transform the matrix to upper Hessenberg form? All elements a_{ij} of an upper Hessenberg matrix are zero for $i > j + 1$. Explain your reasoning.
2. (a) Consider the exponential function e^x . What is its relative condition number? Should it be possible to compute values of this function accurately or is the function very sensitive to small perturbations in the argument x ?
 - (b) Let us attempt to compute e^x by using the expansion

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

It is known from calculus that the series converges for any real value of x .

In practise we would compute a term $a_n := x^n/n!$ in the series by using that $a_n = a_{n-1}x/n$ where a_{n-1} is the previous term. We also stop adding terms when they are so small that the partial sum no longer changes. Also note that we have to use a floating point system.

If we use this method for $x = 5$ or $x = 10$ we obtain quite accurate values for the exponential. However, that is no longer true $x = -5$ and for $x = -10$ we even obtain a negative sum if we use single precision. What accounts for this failure and the large difference between positive and negative x ?

- (c) Can you think of a trick which will save this algorithm so that it also could be used reliably for negative x .
3. (a) Explain how the coefficients of a quadratic polynomial $p_2(x)$ can be determined by linear least squares from data given by a set of at least 4 coordinates (x_i, y_i) where the x_i are all different.
- (b) What would happen if there are only three sets of coordinates? What algorithm would you then use?
4. (a) What is the best approximation of the function x^{n+1} in the maximum norm using polynomials of degree n and working in the interval $[-1, 1]$? What theorem is the foundation of your answer?
- (b) Show that as good an approximation can be obtained for this function if we work with polynomials of degree $n - 1$.
- (c) What is the best approximation of the function x^{n+1} in the maximum norm using polynomials of degree n and working in the interval $[0, 1]$?
5. Consider three-point numerical quadrature rules to approximate $\int_a^b f(x)dx$.
- (a) Assume that we select a and b as two of these points. Can we then make sure that we can integrate all second order polynomials exactly with some quadrature rule of this kind for whatever choice we make for the third point? Assume that the third point differs from a and b .
- (b) Show that we can integrate all cubic polynomials with such a quadrature rule if and only if we place the third point at $(a+b)/2$.
- (c) How large a family of polynomials can we integrate exactly if we are free to select those three points anywhere in the interval?
6. (a) Describe how to construct a piece-wise linear interpolant of a continuous function $f(x)$ given on a bounded interval and using equidistant points.
- (b) What can you say about the error if the distance between consecutive points is h and $f(x)$ is a sufficiently smooth function?
- (c) Can we have a Runge phenomenon? Explain.
- (d) How much work is required to construct such a piece-wise linear interpolant?
- (e) What can be said about the error when using piecewise cubic Hermite interpolation on the same set of points?

7. The Laguerre orthogonal polynomials are defined using the inner product

$$\langle f, g \rangle = \int_0^{\infty} e^{-x} f(x) g(x) dx.$$

- (a) Develop the one point Gaussian quadrature rule for this inner product.
- (b) Develop the two-point Gaussian quadrature rule for this inner product.
- (c) For what families of functions are these quadrature rules exact?
8. (a) Consider a positive, real number, which is on the order of 1. What can then be said about the relative error of its floating point representation if we use IEEE double precision with 52 bits allocated for the numerical value?
- (b) Let a, b , and c be three double precision floating point numbers all of order 1. What can be said, in the worst case, about the relative error of $a * b$, $\sqrt{a * b}$, and $a + b * c$, respectively?
9. Given $n + 1$ distinct real numbers x_0, \dots, x_n and a function $f(x)$ and its first derivative at these points, one can compute its Hermite interpolant

$$p_{2n+1}(x) = \sum_{k=0}^n [H_k(x)f(x_k) + K_k(x)f'(x_k)].$$

This polynomial p_{2n+1} has the same value as $f(x)$ at the x_i and its derivative has the same value as the derivative of $f(x)$ at the same points.

- (a) How are the polynomials $H_k(x)$ and $K_k(x)$ determined by solving special interpolation problems? If you wish you can consider the case of $n = 1$ in your discussion.
- (b) How can one establish the uniqueness of the Hermite interpolation problem by using an elementary property of polynomials?
- (c) In the textbook, the Hermite interpolation formula given above is used to derive Gaussian quadrature rules. These are of the form $A_0 f(x_0) + \dots + A_n f(x_n)$. Somehow the terms involving the values of $f'(x)$ disappear. How does that happen?