Numerical Analysis. Problems from earlier Midterm exams with answers.

1. Consider the equation $x^2 - 2 = 0$.

   (a) Starting with $x_0 = 1$ and $x_1 = 2$ compute x_2 and x_3 using the bisection method.

   (b) Starting from the same x_0 and x_1 compute x_2 and x_3 using the secant method.

   (c) Starting from $x_0 = 1$, compute x_1 and x_2 using Newton's method.

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   (a) We start with the interval $[1, 2]$. This interval then shrinks to $[1, 3/2]$ and then to $[5/4, 3/2]$.

   (b) $x_2 = x_1 - \frac{f(x_1)}{\frac{f(x_1)-f(x_0)}{x_1-x_0}} = 4/3$. $x_3 = 7/5$.

   (c) $x_1 = x_0 - \frac{f(x_0}{f'(x_0)} = 3/2$. Similarly, $x_2 = 17/12$.

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2. Consider a $m \times n$ matrix A.

   (a) Explain how to find a Householder matrix Q such that QA has zero elements in the first column except for its first element.

   (b) What is the form of Q and how do we know it has orthonormal columns?

   (c) Explain how to find a Householder matrix Q such that the 11 element of QA is the same as that of A and such that all elements of the first column of QA except for the first two equal 0.

   (d) Computing the product of an $m \times m$ matrix times a $m \times n$ matrix generally requires $m^2 n$ multiplications. Explain how we can multiply a Householder matrix Q by a $m \times n$ matrix A using much fewer multiplications.

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   See the text book for (a)-(c).
   (d): $Q = I - 2(uu^T)/u^T u)$. Consider Qa where a is a column of A. Then

$Qa = a - 2(u^T a)/(u^T u)u$. We first can compute the scalar $2/(u^T u)$; it is the same scalar for all columns of A. Then we do one inner product $u^T a$ for each column of A and form the linear combination of a and u. The savings are considerable.

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3. Consider a non-singular, tri-diagonal, symmetric matrix A and a linear system $Ax = b$.

   (a) Is it always possible to solve this linear system with Gaussian elimination?

   (b) Is it always possible to solve this linear system with Cholesky's algorithm?

   (c) If one or both of these methods can be used, what is the amount of storage required and how many multiplications are needed?

   (d) Do you know a bound for $\|\delta x\|/\|x\|$ in terms of $\|\delta b\|/\|b\|$ if

$$A(x + \delta x) = b + \delta b?$$

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(a) Any linear system of algebraic equations with a non-singular matrix can be solved by using Gaussian elimination.

(b) The matrix must be positive definite.

(c) If pivoting is required, we need four vectors of length n. Cholesky's method requires only two. The number of operations is a small multiple of n.

(d) See Theorem 2.11 on p. 72 of the text book.

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4. Let A be a $m \times n$ matrix with more rows than columns and consider the least squares problem $\min_x \|Ax - b\|_2$.

   (a) Does this problem always have a solution? Is there a condition which assures us that the solution is unique?

   (b) What is a good method of solving this problem and how is it done using Matlab?

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(a) $\|Ax - b\|_2^2$ is a non-negative function which is a differentiable function of $x_1, \ldots, x_n$. Any such function has a minimum. If x and y are two different vectors which provide the minimum, then $A(x - y) = 0$. This happens if and only if the columns of A are linearly dependent.

(b) Using a method based on a QR factorization of A is more reliable than forming and solving the normal equations. MATLAB uses a QR factorization.

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5. Consider a symmetric, positive definite matrix A with a very special structure, namely, all elements in the first row and the first column differ from zero while all other off-diagonal elements equal zero.

   (a) Show that all elements on and below the diagonal of the Cholesky factor L will be different from zero. (Therefore we do not take much advantage of all the zeros in the matrix A.)

   (b) Find a permutation P so that $P^T AP$ will have a Cholesky factor with a lot of zero elements.

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(a) As far as the loss of zeros, there is no difference between Gaussian elimination and Cholesky's algorithm. It is easy to see that in the first step on Gaussian elimination, all the zeros are replaced by non-zeros. The cost of factoring the matrix is then essentially the same as if all elements in the matrix differed from zero initially.

(b) We can remedy this situation by exchanging the first and last row and at the same time exchanging the first and last columns. Or we can reverse the order of the rows and the order of the columns. No zeros are then lost and the work is linear in n.