Numerical Analysis. Problems from earlier Midterm exams.

1. Consider the equation $x^2 - 2 = 0$.

   (a) Starting with $x_0 = 1$ and $x_1 = 2$ compute x_2 and x_3 using the bisection method.

   (b) Starting from the same x_0 and x_1 compute x_2 and x_3 using the secant method.

   (c) Starting from $x_0 = 1$, compute x_1 and x_2 using Newton's method.

2. Consider a $m \times n$ matrix A.

   (a) Explain how to find a Householder matrix Q such that QA has zero elements in the first column except for its first element.

   (b) What is the form of Q and how do we know it has orthonormal columns?

   (c) Explain how to find a Householder matrix Q such that the 11 element of QA is the same as that of A and such that all elements of the first column of QA except for the first two equal 0.

   (d) Computing the product of an $m \times m$ matrix times a $m \times n$ matrix generally requires $m^2 n$ multiplications. Explain how we can multiply a Householder matrix Q by a $m \times n$ matrix A using much fewer multiplications.

3. Consider a non-singular, tri-diagonal, symmetric matrix A and a linear system $Ax = b$.

   (a) Is it always possible to solve this linear system with Gaussian elimination?

   (b) Is it always possible to solve this linear system with Cholesky's algorithm?

   (c) If one or both of these methods can be used, what is the amount of storage required and how many multiplications are needed?

   (d) Do you know a bound for $\|\delta x\|/\|x\|$ in terms of $\|\delta b\|/\|b\|$ if

   $$A(x + \delta x) = b + \delta b?$$

4. Let A be a $m \times n$ matrix with more rows than columns and consider the least squares problem $\min_x \|Ax - b\|_2$.

   (a) Does this problem always have a solution? Is there a condition which assures us that the solution is unique?

   (b) What is a good method of solving this problem and how is it done using Matlab?

5. Consider a symmetric, positive definite matrix A with a very special structure, namely, all elements in the first row and the first column differ from zero while all other off-diagonal elements equal zero.

   (a) Show that all elements on and below the diagonal of the Cholesky factor L will be different from zero. (Therefore we do not take much advantage of all the zeros in the matrix A.)

   (b) Find a permutation P so that $P^T AP$ will have a Cholesky factor with a lot of zero elements.

6. Consider the equation $x^2 - 3 = 0$.

   (a) Show, starting from $x_0 = 1$, that Newton's method converges to $\sqrt{3}$.

   (b) Name a method which starts with an interval in which a root lies and in each iteration computes a smaller interval containing the root. Outline one such algorithm.

   (c) Is the secant method such an algorithm?

   (d) Starting with $x_0 = 1$ and $x_1 = 2$ compute x_2 and x_3 using the secant method.

7. (a) What is a Householder matrix?

   (b) Discuss, preferably more than one, numerical problem for which such matrices are used.

   (c) Computing the product of a $m \times m$ matrix times a $m \times n$ matrix generally requires $m^2 n$ multiplications. Explain how we can multiply a Householder matrix by a $m \times n$ matrix A using much fewer multiplications.

8. Consider the expression $\|x\|_1 = \sum_{i=1}^{n} |x_i|$, where the x_i are the components of a vector in R^n.

   (a) Show that $\|x\|_1$ is a vector norm.

(b) How do we define the corresponding matrix norm $\|A\|_1$ where A is a $n \times n$ matrix?

(c) How can we compute this matrix norm from the elements a_{ij} of the matrix A?

9. (a) Consider five points in the plane (x_i, y_i), $1 \leq i \leq 5$, where all the x_i are different. Assume that these points almost lie on a curve given by $y(x) = p_2(x)$, where p_2 is a quadratic polynomial. Explain how to determine such a polynomial by solving a linear least squares problem. Form the matrix and the right hand side.

   (b) Can this problem be reduced to a 3×3 linear system of equations? If so, write it down.

   (c) Show that this least squares solution is unique.

   (d) How would you solve this problem using Matlab? What is the underlying algorithm?

10. Explain how we can determine if a symmetric five-diagonal matrix is positive definite or not. Explain how this can be done using on the order of n arithmetic operations if the matrix is $n \times n$.