YOUR NAME:

Numerical Analysis, Midterm, March 31, 2015.

Cross out what is not meant to be part of your answers.

For full credit motivate your answers.

1. (a) What is the purpose of the Sturm sequence algorithm?

   (b) Explain why we can compute the determinant of a symmetric $n \times n$ tridiagonal matrix and the determinants of all its principal minors using only on the order n arithmetic operations. If you write down a formula, explain its origin.

   (c) Suppose you wish to find the largest eigenvalue of a symmetric tridiagonal matrix. Explain how you can find an upper bound using Gerschgorin's theorem.

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(a) The computation of a Sturm sequence, $p_1(\lambda), p_2(\lambda) \ldots, p_n(\lambda)$, provides the sign of the determinant of a symmetric tridiagonal matrix $T - \lambda I$ and all its principal minors. This gives information on how many eigenvalues of T are larger than λ. Using a bisection algorithm, we can then find individual eigenvalues in any interval we might be interested in.

(b) An algorithm is provided on p. 157 of the textbook where there is also a description of how this formula can be derived.

An alternative is provided by using Gaussian elimination noting that the product of the diagonal elements of the upper triangular factor equals the determinant or the negative of the determinant. Pivoting is required which makes the full story a bit complicated.

(c) If a row of the matrix has the three nonzero elements b_i, a_i, b_{i+1} then Gerschgorin's theorem gives the upper bound

$$\max_i a_i + |b_i| + |b_{i+1}|,$$

where this formula needs to be modified for $i = 1$ and $i = n$.

2. Consider a $m \times n$ matrix A.

   (a) Explain how to find a Householder matrix Q such that QA has zero elements in the first column except for its first element.

   (b) What is the form of Q and how do we know it has orthonormal columns?

   (c) Is it possible to find more than one such Q?

   (d) Explain how to find a Householder matrix Q such that the 11 element of QA is the same as that of A and such that all elements of the first column of QA except for the first two equal 0.

   (e) Computing the product of an $m \times m$ matrix times a $m \times n$ matrix generally requires $m^2 n$ multiplications. Explain how we can multiply a Householder matrix Q by a $m \times n$ matrix A using much fewer multiplications.

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A Householder matrix has the form $Q := I - 2\frac{vv^T}{v^T v}$ where v is a column vector with m components.

(a) If the first column of A is the vector $a^{(1)}$, then we choose $v = a^{(1)} - \|a^{(1)}\|_{\ell^2} e_1$ or $a^{(1)} + \|a^{(1)}\|_{\ell^2} e_1$.

(b) A direct computation shows that $Q^2 = I$.

(c) Yes, compare the two formulas above.

(d) Replace the vector v given above by a vector similarly constructed but using the vector with the first element 0 and the second through nth components of $a^{(1)}$.

(e) Compute the vector $v^T A$, form the vector $\frac{-2}{v^T v} v$, then multiply those two vectors to form a $m \times n$ matrix. Add A. Counting multiplications, we find a small multiple of mn.

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3. (a) Given a $m \times n,\ m > n$, matrix A and a m-vector b, explain what the corresponding linear least squares problem is. Under what condition will this problem have a unique solution?

   (b) Explain two different methods to solve such a problem and explain which one is preferred and why.

   (c) How do we most conveniently solve such a problem in MATLAB?

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(a) The least squares solution is the solution of $\min_x \|Ax - b\|_{\ell^2}$, which always exists. The solution is unique if and only if the columns of A are linearly independent.

(b) We can set up and solve the normal equations or solve it by using a QR factorization of A. The second algorithm suffers less from round-off problems and is therefore to be preferred.

(c) A backslash b.

4. (a) Suppose we wish to compute the Lagrange interpolation polynomial of a given function $f(x)$. Assume that we work on the interval $[-1, +1]$. Why is it better to use the Chebyshev points instead of equidistant points if the number of points is large?

   (b) What are the Chebyshev polynomials $T_k(x)$?

   (c) How do we know that $T_k(x),\ k = 0, 1, \ldots, n$ form a basis for $\mathcal{P}_n$, the space of all polynomials of degree n or less.

   (d) Write $1+x+x^2$ as a linear combination of Chebyshev polynomials.

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(a) Runge's phenomenon rules out success with equidistant point. See also the part of the the handout marked Chapter 15, which explains the role of the Lebesgue constants. The Lebesgue constant is almost prefect for the Chebyshev points for which we also can show the convergence of the interpolation polynomials to the given function $f(x)$ when the degree $n \to \infty$ under a very mild assumption on the smoothness of $f(x)$.

(b) $T_k := \cos(k \arccos(x))$.

(c) They satisfy $T_{k+1}(x) = 2xT_k(x) - T_{k-1}$; use a simple trigonometric formula. This shows that the degree of T_k is exactly k. We can write any polynomial $p(x) := a_0 + a_1x + \ldots + a_nx^n$ as a linear combination of $T_k(x), k = 0, 1, \ldots, n$; first find the coefficient b_n such that $p(x) - b_nT_n(x)$ is a polynomial of degree $n - 1$ or less. Repeat to find b_{n-1}, etc.

(d) $T_0 = 1,\ T_1 = x,\ T_2(x) = 2x^2 - 1$. Therefore,

$$x^2 + x + 1 - \frac{1}{2}T_2(x) = x + \frac{3}{2} = T_1(x) + \frac{3}{2}T_0(x).$$

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5. Let A be an arbitrary symmetric tridiagonal matrix.

   (a) Show that $B := A^2 + I$ always is symmetric, positive definite.

   (b) What can we say about the sparsity of B?

   (c) Consider the linear system $Bx = b$. How much storage is needed to solve this system with Cholesky's algorithm?

   (d) Estimate the number of arithmetic operations as a function of n, the order of the matrix A.

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(a) Since A is symmetric, all its eigenvalues λ are real. B is clearly symmetric and its eigenvalues equal $\lambda^2 + 1$. Thus, it is positive definite.

(b) The square of a tridiagonal matrix is penta-diagonal, i.e.,

$$b_{ij} = 0, \quad \forall i, j, \text{ such that } |i - j| > 2.$$

This is easy to see by a direct computation.

(c) Storage required for the matrix, and then for its Cholesky factor L, is $3n$.

(d) Using the sparsity, we see that only the first three rows are involved when we compute the first column of L. Only 6 elements are involved. The matrix that remains to factor after the first step is also penta-diagonal. The number of multiplications and additions required is therfore bounded by a small multiple of n.