Assignment Set 1, MATH-UA0252-1

Olof Widlund

February 3, 2015

The following assignments are due on Wednesday February 18 at 12:00 noon. No homework will be accepted after that time. You should give me your homework in class or put it under my office door (WWH612). Do not use my mailbox in the WWH lobby.

You should turn in your MATLAB/OCTAVE programs and print-outs which shows how they work for different functions. In addition, is is important that you comment on what you can learn about the performance of the methods; it is appropriate to count the number of function evaluations to arrive at a good approximation of the roots. An error of 10^{-6} will be appropriate for your experiments but it also makes sense to find out what maximum accuracy can be obtained. Always use format long e in MATLAB. Note that in MATLAB, all numbers are represented with approximately 16 decimals; we will discuss computer arithmetic in the middle of the term.

Problem: Find roots of $f(x) = 0$ where $f(x)$ is a continuous function.

1. Write a procedure that implements the bisection method to find a root of $f(x) = 0$. Note that two initial values x_0 and x_1 with $f(x_0)f(x_1) \leq 0$ are required. In some cases, plotting the function using MATLAB is a good idea for finding initial values.

2. Write a procedure that implements the secant method to find a root of $f(x) = 0$. Two initial values are required but we do not need a change of sign of $f(x)$ between them.

3. Write a procedure that implements Newton's method to find a root of $f(x) = 0$. One initial value is needed as well as a second subprogram to evaluate $f'(x)$.

4. Write a procedure that implements the "Illinois" method to find a root of $f(x) = 0$. Here is a description of this method.

   We use the notation $f_n := f(x_n)$. We start with two points x_0 and x_1 for which f has opposite sign. Now assume that for a certain n, $f_{n-1}f_n < 0$. Then x_{n+1} is defined by the secant approximation. If $f_n f_{n+1} < 0$, we can proceed to the next step using x_n and x_{n+1} unless $|x_{n+1} - x_n|$ is small enough. Otherwise, a modified formula is used: In this second case, there is a change of sign between x_{n-1} and x_{n+1}, since $f_{n-1}f_n < 0$ and $f_n f_{n+1} > 0$. Find the intersection of the straight line through $(x_{n+1}, f(x_{n+1}))$ and $(x_{n-1}, f(x_{n-1})/2)$ with the $x - axis$. This point is chosen as x_{n+2} if there is a change of sign between it and x_{n+1}. Otherwise, find the intersection of the straight line through $(x_{n+1}, f(x_{n+1}))$ and $(x_{n-1}, f(x_{n-1})/4)$ with the $x - axis$, and test for a change of sign. If necessary, additional points are computed and tested after replacing $f(x_{n-1})/4$ by $f(x_{n-1})/8$, etc. (It can be established that that we eventually get a sign change and that therefore the algorithm never gets stuck in an infinite loop.) Note that once we are successful, the assumption $f_{n+1}f_{n+2} < 0$ again is valid and we can proceed using x_{n+1} and x_{n+2}.. If necessary, i.e., when the two most recent points are not sufficiently close to each other, we continue the iteration, using the same recipe.

All your programs should be written so that, for any method, the function f is never computed twice for the same argument.

After trying your programs on some simple functions, e.g. $x^2 - 2 = 0$, use them all to find an accurate root of

$$f(x) = \sin(x^2) + 1.02 - \exp(-x) = 0.$$

Its derivative is

$$f'(x) = 2x\cos(x^2) + \exp(-x) \; .$$

If there is more than one root, first find the *one furthest to the left.* In order to get started, make an approximate plot of the function, e.g., by using a MATLAB graphics tool to plot close-ups of the function. Then find as many additional roots as you can. Is there any difference in the accuracy with which you can find the different roots?

Compare the performance of the four methods. How many steps are required for each to get the best accuracy possible; there is no point to continue an iteration if $x_{k+1} = x_k$. How many steps are required to reach an error below 10^{-6}?

Also try to find a function $f(x)$ and starting values for which the methods in 2 and 3 do not converge.

Your score will be determined primarily by the quality of your written discussion of the results, with *selected* program listings and computer output as supporting evidence. Also provide hard copy graphics output.

5. Problem 1.3 from page 35 of the text book.

6. Problem 1.8 on page 37 of the text book.