PDE in Finance

Announcement
Every homework is due 2 weeks after each class. The first 2 homework should be turned in by Feb. 22. You may put them in Junyoeo Park’s mail box in the lobby of math building.

Lecture 3. Due Feb.28

Exercise 1. Let a stochastic process $x_t$ solve the following Ornstein-Uhlenbeck equation:

$$dx = -kx dt + \sigma dW_t, \quad x(0) = x_0$$

where $k, \sigma > 0$.
(a) Find the SDE that $y = e^{kt}x$ solves and by using it, find the explicit formula for $x(t)$.
(b) From (a), compute the mean $E(x(t))$ and variance $Var(x(t))$.
(c) Write down the PDE which is associated with the conditional expectation $\phi(x,t) = E(e^{\lambda x(t)}|x(t) = x)$. Verify that the moment generating function of a Gaussian random variable with mean and variance from (b) solves that PDE. What can you conclude about the distribution of $x(t)$?

Exercise 2. Feller process is a stochastic process that satisfies the following SDE:

$$dr = k(\theta - r)dt + \sigma \sqrt{r}dW_t, \quad r(0) = r_0$$

where $k, \sigma > 0$, and $k\theta > \frac{1}{2}\sigma^2$.
(a) Write down the PDE which is associated with the moment generating function $\phi(x,t) = E(e^{-\lambda r(t)}|r(t) = x)$.
(b) A solution of above final-value problem can be sought in the form

$$\phi(x,t,T;\lambda) = \exp(-f(t,T;\lambda)x - g(t,T;\lambda))$$

By substitution of this function into the PDE, verify that $f$ and $g$ solve the following ODE

$$\dot{f} - kf = \frac{1}{2}\sigma^2 f^2, \quad \dot{g} + k\theta f = 0$$

$$f(T,T;\lambda) = \lambda, \quad g(T,T;\lambda) = 0$$

(c) The first ODE in (b) is Ricatti equation and it can be solved in closed form. Solve the two ODEs in (b).
Hint. Divide the first equation by $f^2$ and then obtain a linear ODE in terms of $\frac{1}{f}$. 
We find, after some calculation, that
\[
f(t, T; \lambda) = \lambda e^{-k(T-t)} \frac{1}{1 + \frac{\lambda \sigma^2}{2k} (1 - e^{-k(T-t)})}
\]
\[
g(t, T; \lambda) = \frac{2k \theta}{\sigma^2} \log \left( 1 + \frac{\lambda \sigma^2}{2k} (1 - e^{-kT}) \right)
\]

Setting \( t=0 \) and \( T=t \), this implies that the moment-generating function of \( r(t) \) is that of the non-central \( \chi^2 \) distribution. And it is well-defined for all parameter values such that \( 0 < \frac{k \theta}{\sigma^2} < 0.5 \).

Exercise 3. Let a stochastic process \( x_t \) solve the following SDE:
\[
dx = x_t dW_t + \beta x_t^{\gamma+1} dt, \quad x(0) = x > 0, \text{ and } \gamma > 0
\]
(a) Define \( z_t = \log(x_t) \) and find an SDE that \( z_t \) solves.
(b) Using result in (a) show that \( x_t \) satisfies the following equation
\[
x(t) = x(0) M(t) \exp(\beta \int_0^t x^\gamma ds)
\]
where \( M(t) = \exp(W_t - \frac{1}{2} t) \) is the exponential martingale.
(c) Define \( y(t) = \int_0^t x^\gamma ds \). Deduce an ODE that \( y(t) \) solves and compute it. Find a closed solution of \( x(t) \).