PDE in Finance

Lecture 2.

Exercise 1. The moment generating function \( \phi(k) \) of a random variable \( X \) is defined by:

\[
\phi(k) = \int e^{ikX} P(dX)
\]

a. Compute the moment generating function of a Brownian Motion \( W_t \) and find the first four moments of the Brownian Motion.

Hint) Verify that \( \phi^{(n)}(0) = i^n E(W^n) \).

b. Compute \( E(e^{\lambda W_t}) \) for arbitrary \( \lambda \).

c. Define \( M_\lambda(t) = \exp(\lambda W_t - \lambda^2 t/2) \). Deduce from the calculation of b that \( M_\lambda(t) \) is a martingale and discuss the SDE associated with \( M_\lambda \). This \( M_\lambda \) is a lognormal process, and with \( \lambda = \sigma \) it is used in the BS option pricing formula.

Exercise 2. Hermite polynomial \( H_n(x) \) of order \( n \) is the function defined by:

\[
H_n(x) = \frac{1}{n!} e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}
\]

The Taylor expansion of \( e^{-(x+y)^2/2} \) at \( x = 0 \) fixing \( y \) as a constant yields

\[
\exp\left(-\frac{(x + y)^2}{2}\right) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \left(\frac{d}{dy}\right)^n e^{-\frac{y^2}{2}}
\]

Then,

\[
\exp\left(-\frac{x^2}{2} - xy\right) = \sum_{n=0}^{\infty} x^n \frac{e^{\frac{y^2}{2}}}{n!} \left(\frac{d}{dy}\right)^n e^{-\frac{y^2}{2}} = \sum_{n=0}^{\infty} x^n H_n(y)
\]

Take \( x = -\lambda \sqrt{t} \) and \( y = \frac{W_t}{\sqrt{t}} \). Then we note

\[
M_\lambda(t) = \exp(\lambda W_t - \lambda^2 t/2) = \sum_{n=0}^{\infty} (-1)^n t^{n/2} H_n\left(\frac{W_t}{\sqrt{t}}\right) \lambda^n
\]

where \( M_\lambda \) is from part c of Exercise 1.

Define \( \mu_n(t) \) iteratively by

\[
\mu_n(t) = \left(\frac{d}{d\lambda}\right)^n M_\lambda(t) |_{\lambda=0}
\]
a. Compute $\mu_1, \mu_2, \mu_3, \mu_4$ and show that they are martingales.
b. Show $\mu_n(t) = (-1)^n n! n/2 H_n(W_t/\sqrt{t})$.

Exercise 3 (Option pricing with volatility skew). Attach the computing code with your answer sheets. Assume that the risk free rate is zero and the dividend rate is zero. And the risk neutral stock price process solves the following SDE:

$$
dS = \sigma(S, t)dW_t
$$

$$
\sigma(S, t) = \sigma_0 + \Phi(5 \log(S_t/S_0))(\sigma_1 - \sigma_0)
$$

where $\Phi(x)$ is the cumulative standard normal distribution. We have the initial parameters $\sigma_0 = 0.4, \sigma_1 = 0.2, S_0 = 100$.
a. Price a European call option on $S$ with strike $K = 110$ and maturity $T = 90$ days from now using the trinomial finite-difference scheme.
b. Compute the delta $\frac{\partial C}{\partial S}$ and gamma $\frac{\partial^2 C}{\partial S^2}$. 
