PDE in Finance

Lecture 1.

Solving some linear PDE’s by Fourier Transform.

For a smooth function \( u(x) \) which decays rapidly as \( x \to \infty \), the Fourier transform of \( u \) is the function defined by

\[
F u(k) \equiv \hat{u} \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixk} u(x) dx
\]

And the Inverse Fourier Transform is defined by

\[
F^{-1} u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixk} \hat{u}(k) dk
\]

It is known that \( F^{-1} F(u) = u \).

Exercise 1. Compute the Fourier Transform of the following functions

(a) \( f(x) = \frac{1}{\sqrt{2\pi\epsilon}} \exp(-x^2/(2\epsilon^2)) \).
(b) \( f(x) = \delta_a \) where \( \delta_a \) is the dirac mass at point \( a \).
(c) \( f(x) = \exp(-c\|x\|), c > 0 \) is a constant.

Exercise 2. Write the Fourier Transform of the following functions in terms of \( \hat{u} \) and \( \hat{v} \).

(a) \( u(x-\lambda) \)  
(b) \( e^{i\lambda x} u \)  
(c) \( u(\lambda x) \)  
(d) \( \frac{d^n u}{dx^n} \)

(c) \( u * v(x) = \int u(y)v(x-y)dy \), \( u * v \) is called the convolution of \( u \) and \( v \).

The characteristic function, shortly Chf, of a random variable \( X \) is defined by

\[
\phi(k) = \int_R e^{ikx} P(dx)
\]

where \( P \) is the law of the random variable \( X \). Then we see that the Chf is closely related to the Fourier transform. Especially, if the random variable has a density function \( f(x) \), then the Chf of \( X \) is nothing but the inverse Fourier transform of \( f \) up to scaling.

Exercise 3. Show that if \( X \) and \( Y \) are two independent random variables taking values in \( R \), Chf of \( X+Y \) is the product of Chf of \( X \) and Chf of \( Y \). (Hint, show that the law of \( X+Y \) is the convolution of the law of \( X \) and the law of \( Y \), and apply part (c) in Exercise 1.)

Exercise 4(Central Limit Theorem). Let \( X_i, i = 1, 2, \ldots \) be the sequence of independent and identically distributed random variables on real \( R \) with mean 0 and variance 1. Assume that the Chf of \( X_i \) is in \( C^3 \) with bounded 3rd derivative. Using Exercise 3, show that the characteristic function of
\[ S_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \] converges to the Chf of the standard Gaussian distribution. What is your conclusion?

Assume that \( u \) solves the following PDE.

\[
\begin{align*}
    u_t - \frac{\sigma^2}{2} u_{xx} &= 0, \quad t > 0 \\
    u(0, x) &= f(x)
\end{align*}
\]

This PDE is called diffusion equation.

First fix \( t \) variable as a constant, and take the Fourier Transform of the equation with respect to \( x \) variable. Then we have

\[
\begin{align*}
    \hat{u}_t + \frac{\sigma^2}{2} k^2 \hat{u} &= 0 \\
    \hat{u}(0) &= \hat{f}
\end{align*}
\]

The second term of the first equation is justified by (d) in Exercise 2. The Fourier Transformed equation can be seen as a simple linear ODE in \( t \) variable with the initial condition as \( \hat{u}(0) = \hat{f} \). It is solvable by method of separation of variables, and \( \hat{u} \) satisfies the following explicit formula.

\[
\hat{u}(t, k) = \hat{f}(k) e^{-\frac{k^2}{2} \sigma^2 t}
\]

By the Fourier Inversion Formula \( F^* F = \text{id} \), then we recover \( u \) by taking \( F^* \) on \( \hat{u} \). Therefore, the solution \( u \) for the diffusion equation is the following:

\[
\begin{align*}
    u(t, x) &= F^* \hat{u} \\
    &= \frac{1}{\sqrt{2\pi}} \int e^{ixk} \hat{f}(k) e^{-\frac{k^2}{2} \sigma^2 t} dk \\
    &= \frac{1}{\sqrt{2\pi}} \int e^{ixk} - \frac{\sigma^2}{2} k^2 t \frac{1}{\sqrt{2\pi}} \int e^{-iky} f(y) dy dk \\
    &= \frac{1}{2\pi} \int \int e^{ixk} - \frac{\sigma^2}{2} k^2 t - iky f(y) dy dk \\
    &= \frac{1}{\sqrt{2\pi \sigma^2 t}} \int f(y) e^{-\frac{(x-y)^2}{2\sigma^2 t}} dy
\end{align*}
\]

The last step is justified by the change of variables, the Fubini Theorem of the change of order of integrations and the Contour Integral of the Gaussian function. The function \( K(t, y) = \frac{1}{\sqrt{2\pi \sigma^2 t}} \exp(-\frac{(x-y)^2}{2\sigma^2 t}) \) is called diffusion kernel which is closely related to the density of diffusion process. As \( t \to 0 \), \( K(t, y) \) converges to \( \delta_x \), the dirac-delta function at \( x \).

Exercise 5. Justify the last step of the above calculation in detail.

Now we look at the condition that \( t > 0 \). Assume that \( t \) can be negative, then do the formal calculation just as what we did so far. The formula for \( u \) still
holds at least formally. Take a very small \( t \approx 0 \) but still negative. The exponent in the integration becomes very large and hence the integration gets unstable, which means a small change of the initial condition may cause a big change in the answer. Such a problem is called 'ill-posed'. This is why we need the condition \( t > 0 \).

Exercise 6. Solve the following backward diffusion equation. Be aware that in this problem, we have the positive sign in front of \( u_{xx} \) and we have the final value condition at \( \{ t = T \} \).

\[
\begin{align*}
    u_t + u_{xx} &= 0, \quad 0 < t < T \\
    u(T, x) &= e^{-x^2}
\end{align*}
\]

Exercise 7. The solution \( u(x,t) \) of the diffusion equation can be thought as the convolution of the initial function \( f(x) \) with a Gaussian. Assume that \( f \) is \( n \) times differentiable with bounded and Holder continuous \( n \)-th derivative. Then show that the same is true for the solution \( u(x,t) \) of the diffusion equation. Hint. See the following proof for \( n = 0 \).

Let \( f \) be such that \(| f(x) - f(y) | < C |x - y| \) for some constant \( C \) and for all \( x, y \). Let \( K(t,x) \) be the diffusion kernel for diffusion equation. Then \( u(t,x) = (K * f)(x) = \int K(t,x - y)f(y)dy \). For arbitrary \( x \) and \( y \), we have the following inequalities.

\[
|u(t,x) - u(t,y)| = |\int K(t,x - z)f(z) - K(t,y - z)f(z)dz|
\]

\[
= |\int K(t,z)(f(z - x) - f(z - y))dz|
\]

\[
\leq \int K(t,z)|f(z - x) - f(z - y)|dz
\]

\[
\leq C|x - y|\int K(t,z)dz
\]

\[
= C|x - y|, \quad \text{since} \int K(t,z)dz = 1
\]

Above inequality shows that \( u \) is Holder continuous for \( n = 0 \).

The calculation of PDE by Fourier Transform can be generalized for more complicated equations. The \( d \)-dimensional Fourier and Inverse Fourier Transform are defined by

\[
\begin{align*}
    F u(k) &= \left(\frac{1}{2\pi}\right)^{d/2} \int_{\mathbb{R}^d} e^{-i<k,x>} u(x)dx \\
    F^* u(x) &= \left(\frac{1}{2\pi}\right)^{d/2} \int_{\mathbb{R}^d} e^{i<k,x>} u(k)dk
\end{align*}
\]

where \(<,>\) is the inner product of two vectors in \( \mathbb{R}^d \). Again, we have the Fourier Inversion Formula \( F^* F = id \).
Assume $u$ solves the following parabolic PDE.

$$
\begin{align*}
  u_t - \sum_{i,j=1}^{d} a_{ij}(t) u_{x_i x_j} + b(t) u &= 0, \quad t > 0 \\
  u(0, x) &= f(x)
\end{align*}
$$

where $a_{ij}(t), b(t)$ are smooth and bounded functions of $t$ such that $A(t) = (a_{ij}(t))$ is positive definite for all $t$.

To solve above PDE, we take the Fourier Transform with respect to $x$. Then the transformed equation is as following

$$
\hat{u}_t + \sum_{i,j=1}^{d} a_{ij}(t) k_i k_j \hat{u} + b(t) \hat{u} = 0
$$

$$
\hat{u}(0) = \hat{f}
$$

This is an initial value problem of linear ODE which is easily solvable. The explicit formula for $\hat{u}$ is as following

$$
\hat{u}(t) = \exp(- \sum a_{ij}(t) k_i k_j - b(t)) \hat{f}
$$

Then we obtain $u$ by taking inverse Fourier of $\hat{u}$.

$$
u(x, t) = \left( \frac{1}{2\pi} \right)^{d/2} \int_{\mathbb{R}^d} e^{i<x,k>} \hat{u}(k, t) dk
$$

Exercise 8. Solve the following PDE, $x \in \mathbb{R}^3$

$$
\begin{align*}
  u_t - \frac{1}{2} \Delta u + tu &= 0, \quad t > 0 \\
  u(0, x) &= \left( \frac{1}{2\pi} \right)^{3/2} \exp(- \|x\|^2 / 2)
\end{align*}
$$

Fourier method can be applied to other non-parabolic PDE’s. Now, consider the following PDE.

$$
\begin{align*}
  u_{xx} + u_{yy} &= 0 \\
  u(x, 0) &= f(x)
\end{align*}
$$

This is called Laplace’s equation with boundary value problem. To solve this problem, take the Fourier transform of the PDE with respect to $x$ fixing $y$ as a constant. Then we have

$$
\begin{align*}
  -k^2 \hat{u} + \hat{u}_{yy} &= 0 \\
  \hat{u}(0) &= \hat{f}
\end{align*}
$$

Thus, this is a second order linear ODE in $y$ variable. Then we have for $\hat{u}$

$$
\hat{u} = \alpha e^{ky} + \beta e^{-ky}
$$
One possible solution which decays exponentially and satisfies the initial condition is
\[ \hat{u} = \hat{f} e^{-|k|y} \]

Then taking the inverse Fourier gives rise to
\[ u(x, y) = \int e^{ikx} \hat{f} e^{-|k|y} dk \cdots (\star) \]

Exercise 9. Calculate the integration in the (\star) and find a function \( P(x, y) \) such that
\[ u(x, y) = \int P(x, y)f(y)dy \]
This \( P(x, y) \) is the Poisson kernel for Laplace equation.