Jan 1996 Advanced Calculus

Problem 1

Now we know that

\[ \int \int_S f(ax + by + cz) dS = \text{Area}(S) \cdot \left( \text{average of } f(ax + by + cz) \text{ on } (x,y,z) \in S \right) \]

\[ = 4\pi \left( \text{average of } f(ax + by + cz) \text{ on } (x,y,z) \in S \right) \]

Now let's see what the domain is for \( f(ax + by + cz) \), where \( x^2 + y^2 + z^2 = 1 \). Let \( h(x,y,z) = ax + by + cz \) and \( g(x,y,z) = x^2 + y^2 + z^2 = 1 \). Then by the Lagrange Multiplier theorem, we have \( \nabla h = \lambda \nabla g \)

\[ (a, b, c) = \lambda (2x, 2y, 2z) \Rightarrow \frac{a}{2x} = \frac{b}{2y} = \frac{c}{2z} \]

So let \( x = t \), and we have \( y = \left( \frac{b}{a} \right) t \) and \( z = \left( \frac{c}{a} \right) t \). Hence

\[ t^2 + \frac{b^2}{a^2} t^2 + \frac{c^2}{a^2} t^2 = 1 \Rightarrow t = \frac{\pm a}{\sqrt{a^2 + b^2 + c^2}} \]

Therefore

\[ x = \frac{\pm a}{\sqrt{a^2 + b^2 + c^2}} \]
\[ y = \frac{\pm b}{\sqrt{a^2 + b^2 + c^2}} \]
\[ z = \frac{\pm c}{\sqrt{a^2 + b^2 + c^2}} \]

Hence the minimum is

\[ h_{\text{min}} = -\frac{a^2}{\sqrt{a^2 + b^2 + c^2}} + -\frac{b^2}{\sqrt{a^2 + b^2 + c^2}} + -\frac{c^2}{\sqrt{a^2 + b^2 + c^2}} = -\sqrt{a^2 + b^2 + c^2} \]

and the maximum is

\[ h_{\text{max}} = \frac{a^2}{\sqrt{a^2 + b^2 + c^2}} + \frac{b^2}{\sqrt{a^2 + b^2 + c^2}} + \frac{c^2}{\sqrt{a^2 + b^2 + c^2}} = \sqrt{a^2 + b^2 + c^2} \]

Therefore the average of \( f(ax + by + cz) \) on \( (x,y,z) \in S \) is

\[ -1 \]
\[\int_{-\sqrt{a^2+b^2+c^2}}^{\sqrt{a^2+b^2+c^2}} f(t)dt = \int_{-1}^{1} f(t\sqrt{a^2+b^2+c^2})dt\]

Hence

\[
\int \int_S f(ax + by + cz) dS = 4\pi \int_{-1}^{1} f(t\sqrt{a^2+b^2+c^2})dt = 2\pi \int_{-1}^{1} f(t\sqrt{a^2+b^2+c^2})dt
\]

**Workshop method:** We know that \( v = \frac{1}{r} = \frac{1}{|x-x'|} \) is harmonic, i.e. \( \Delta v = 0 \). Now let \( B_\epsilon \) be the punctured sphere at \( x \) with radius \( \epsilon \). Then

\[d(\Omega) = S + \lim_{\epsilon \to 0} d(B_\epsilon)\]

Then we have

\[
\int_{\Omega} (u\Delta v - v\Delta u) dV = \int_S \left( u \frac{dv}{dn} - v \frac{du}{dn} \right) dS + \lim_{\epsilon \to 0} \int_{d(B_\epsilon)} \left( u \frac{dv}{dn} - v \frac{du}{dn} \right) dS
\]

Since \( u \) and \( v \) are harmonic, we know that it equals zero. Hence

\[
\int_S \left( u \frac{dv}{dn} - v \frac{du}{dn} \right) dS = -\lim_{\epsilon \to 0} \int_{d(B_\epsilon)} \left( u \frac{dv}{dn} - v \frac{du}{dn} \right) dS
\]

Now

\[
\nabla v = -\frac{1}{|x-x'|^2} \cdot \hat{r} \quad \text{where} \quad \hat{r} = \frac{x-x'}{|x-x'|}
\]

Then \( \frac{dv}{dn} = -\frac{1}{|x-x'|^2} \cos \beta \). So on \( d(B_\epsilon) \), \( \hat{n} = -\hat{r} \)

\[
\int_{d(B_\epsilon)} u(x') \frac{-1}{|x-x'|^2} \cdot \hat{r} \cdot d\hat{n} = \int_{d(B_\epsilon)} u(x') \frac{1}{\epsilon^2} dS - \int_{d(B_\epsilon)} \frac{1}{\epsilon} \nabla v dS
\]

Since \( u \) is harmonic, \( |\nabla u| < m \). Hence

\[\left| \int_{d(B_\epsilon)} \frac{1}{\epsilon} \nabla u d\hat{n} \right| \leq 4\pi \epsilon^2 \frac{m}{\epsilon} \to 0\]

as \( \epsilon \to 0 \), and

\[
\int_{d(B_\epsilon)} u(x') \frac{1}{\epsilon^2} d\hat{n} \to 4\pi u(x)
\]

Since \( u \) is continuous.
**Problem 3**

Notice

\[
\int_0^1 x^{-x} dx = \int_0^1 e^{-\log x} dx = \int_0^1 \sum_{n=0}^{\infty} \frac{(-x \log x)^n}{n!} dx = \sum_{n=0}^{\infty} \int_0^1 \frac{(x \log x)^n}{n!} dx
\]

\[= \sum_{n=0}^{\infty} \frac{1}{n!} (-1)^n \int_0^1 x^n (\log x)^n dx\]

Now notice that by performing \(u\)-substitution, we have \(u = -(n+1) \log x\), \(du = -(n+1)/x dx\)

\[
\int_0^1 x^n (\log x)^n dx = \int_0^0 x^{n+1} \left( \frac{u}{-(n+1)} \right)^n du = \frac{(-1)^n}{(n+1)^{n+1}} \int_0^\infty e^{-u} u^n du
\]

and recall the gamma function

\[\Gamma(x) = \int_0^\infty e^{-u} u^{x-1} du\]

so

\[\Gamma(n+1) = \int_0^\infty u^n e^{-u} du = n!\]

recall the other properties of the gamma function

\[\Gamma(x + 1) = x\Gamma(x)\]

\[\Gamma(1/2) = \sqrt{\pi}\]

\[\Gamma(n + 1/2) = \frac{(2n)!\sqrt{\pi}}{4^n n!}\]

So

\[
\int_0^1 x^{-x} dx = \sum_{n=0}^{\infty} \frac{1}{n!} (-1)^n (-1)^n \frac{1}{(n+1)^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{(n+1)^{n+1}} = \sum_{n=1}^{\infty} \frac{1}{n^n}
\]

**Problem 4.a**

Now we know that

\[
\frac{dv}{dn} = \nabla v \cdot n \quad \text{and} \quad \frac{du}{dn} = \nabla u \cdot n
\]

Thus we have

\[
\int_S d\frac{dv}{dn} - v \frac{du}{dn} dS = \int_S u \nabla v \cdot n - v \nabla \cdot n dS = \int_S (u \nabla v - v \nabla u) \cdot n dS
\]

and by the Divergence theorem, we have

-3
\[
\int_{\Omega} \nabla \cdot (u \nabla v - v \nabla u) dV = \int_{\Omega} u \Delta v - v \Delta u dV
\]

**Problem 4.b**

Now let \( v = 1/|x - x'| = 1/r \). Then

\[
\frac{dv}{dn} = \nabla v \cdot n = ||\nabla v|| \cdot ||n|| \cos \beta
\]

and since \(||n|| = 1\), we have

\[
\frac{dv}{dn} = ||\nabla v|| \cos \beta = \left| \frac{d}{dr} \frac{1}{r} \cdot r_u \right| = \frac{1}{r^2}
\]

Hence

\[
\frac{1}{4\pi} \int_{x' \in S} u(x') \frac{\cos \beta}{|x - x'|^2} + \frac{1}{|x - x'|} \frac{du}{dn}(x') dS(x') = \frac{1}{4\pi} \int_{x' \in S} u(x') \frac{dv}{dn} + v \frac{du}{dn}(x') dS(x')
\]

then by part (a) we have

\[
= \frac{1}{4\pi} \int_{\Omega} u \Delta v - v \Delta u dV
\]

Since \( \Delta u = 0 \), we have

\[
= \frac{1}{4\pi} \int_{\Omega} u \nabla v dV
\]

Now notice that in any volume

\[
\int_{V} \frac{1}{r} dV = \int_{V} \nabla \cdot \frac{1}{r} dV = \int_{S} \frac{1}{r} \cdot n dS
\]

\[
= \int_{S} \frac{d}{dr} \frac{1}{r} \cdot n dS = \int_{S} \frac{1}{r^2} dS = \frac{4\pi R^2}{r^2}
\]

where \( R \) is the radius of the small sphere we are integrating around. So as \( r > 0 \) and \( R \to 0 \), the integral goes to zero. And if \( r = R \) and \( R \to 0 \), the integral goes to \(-4\pi\). Hence

\[
\Delta \frac{1}{|x - x'|} = -4\pi \delta(x - x')
\]

Hence

\[
= \frac{1}{4\pi} \int_{x' \in \Omega} u(x') \nabla \frac{1}{|x - x'|} dV(x') = \frac{1}{4\pi} \int_{x' \in \Omega} u(x') (-4\pi \delta(x - x')) dV(x') = u(x)
\]
Problem 5.a
For \( x \in [0, 1] \), we have

\[
f(x) = \sum_{n=0}^{\lfloor \log(1/x) \rfloor} 2^n x + \sum_{n=\lfloor \log(1/x)+1 \rfloor}^{\log(2/x)} 2 - 2^n x + 0 < \infty
\]

Thus it converges pointwise.

Problem 5.b
No. Let \( \epsilon = 2^{-k} \) for \( k \in \mathbb{Z} \). Then notice \( f(0) = \sum h(0) = 0 \). But

\[
\left| f(2^{-k}) \right| = \left| \sum_{n=0}^{\infty} h(2^n 2^{-k}) \right| \geq h(1) = 1
\]

Thus it is not continuous at \( x = 0 \).

Problem 5.c
No. Assume that \( f \) did converge uniformly. Since \( f_k(x) = \sum_{n=0}^{k} h(2^n x) \) is continuous for all \( k \), that would imply that \( f \) is continuous. However as we have seen in part (b), \( f \) is not continuous, and thus we have a contradiction.

Problem 5.d
Notice that

\[
h(x) = \begin{cases} 
0 & x = 0 \\
2 - x & x \in (0, 2]
\end{cases}
\]

Hence for all \( \epsilon > 0 \) \( \exists N \) such that

\[
\left| \sum_{n=N}^{\infty} \frac{1}{n} h(2^n x) \right| \leq \frac{1}{N} \left| \sum_{n=N}^{\infty} h(2^n x) \right| \leq \frac{2}{N} < \epsilon
\]

Jan 1996 Complex Variables

Problem 1.a
CASE 1: if \( p \) is in region I, then by the residue theorem, we have

\[
\oint_C \frac{dz}{z - p} = 2\pi i
\]
**CASE 2:** if \( p \) is in region II, then by the residue theorem, we have
\[
\oint_{C} \frac{dz}{z - p} = \oint_{C_1} \frac{dz}{z - p} + \oint_{C_2} \frac{dz}{z - p} = 4\pi i
\]
where \( C_1 \) is the outer circle and \( C_2 \) is in the inner circle.

**CASE 3:** if \( p \) is in region III, then by the residue theorem, we have
\[
\oint_{C} \frac{dz}{z - p} = 2\pi i
\]

**CASE 4:** if \( p \) is in region IV we have
\[
\oint_{C} \frac{dz}{z - p} = 0
\]

Problem 1.b
Notice for \( y_0 \in \mathbb{R} \), by the residue theorem, we have
\[
\int_{\gamma} f(z)dz = 0
\]
So
\[
\int_{-R}^{R} f(z)dz + \int_{0}^{y_0} e^{i(R+iy)-(R+iy)^2}idy + \int_{R}^{-R} e^{i(R+iy)-(R+iy)^2}idy + \int_{y_0}^{0} e^{i(-R+iy)-(-R+iy)^2}dy
\]
and notice
\[
\left| \int_{0}^{y_0} e^{i(R+iy)-(R+iy)^2}idy \right| \leq y_0e^{-R^2}M \to 0
\]
likewise
\[
\left| \int_{y_0}^{0} e^{i(-R+iy)-(-R+iy)^2}dy \right| \to 0
\]
as \( R \to \infty \). Therefore
\[
\int_{-\infty}^{\infty} e^{iz}e^{-z^2}dz = \int_{-\infty}^{\infty} e^{i(x+iy_0)-(x+iy_0)^2}dx
\]

Problem 3.a
Notice
\[
\lim_{z \to 0} \frac{\cot z - \frac{1}{z}}{z} = \lim_{z \to 0} \frac{\cos z - \sin z}{z \sin z} = \left( -\frac{z^3}{3!} + \frac{z^5}{5!} \pm \cdots - \frac{z^3}{3!} + \frac{z^5}{5!} \pm \cdots \right) - \frac{1}{z^2} \left( 1 - \frac{z^2}{3!} + \frac{z^4}{5!} \pm \cdots \right) \to 0
\]
Problem 3.b
So by solving

\[
\left( -\frac{z^3}{2!} + \frac{z^5}{4!} \pm \cdots \right) - \left( \frac{z^3}{3!} + \frac{z^5}{5!} \pm \cdots \right) = (a_0 + a_1 z + a_2 z^2 + \cdots) \left( \frac{z^2}{3!} + \frac{z^6}{5!} - \cdots \right)
\]

Then we get \( a_0 = 0 \), \( a_1 = -1/3 \), and \( a_2 = (1/4! - 1/5!)/(1 + 1/18) \).

Problem 3.c
It converges for all \( z \), since the function is analytic. Hence the radius of convergence is \( \infty \).

Problem 3.d
Recall the infinite product representation of \( \sin z \) is

\[
\sin z = z \prod_{k=1}^{\infty} \left( 1 - \frac{z^2}{\pi^2 k^2} \right) = z \prod_{k=-\infty, k\neq 0}^{\infty} \left( 1 - \frac{z}{\pi k} \right)
\]

Then

\[
\log(\sin z) = \log z + \sum_{k=-\infty, k\neq 0}^{\infty} \log \left( 1 - \frac{z}{\pi k} \right)
\]

using the principle branch cut. Thus taking the derivatives, we have

\[
\frac{1}{\sin z} \cos z = \frac{1}{z} + \sum_{k=-\infty, k\neq 0}^{\infty} \frac{1}{1 - \frac{z}{\pi k}} \left( -\frac{1}{\pi k} \right)
\]

Hence

\[
\cot z = \frac{1}{z} + \sum_{k=-\infty, k\neq 0}^{\infty} \frac{1}{z - \pi k}
\]

so

\[
f(z) = \sum_{k=-\infty, k\neq 0}^{\infty} \frac{1}{z - \pi k}
\]

Problem 4.a
We want to solve \( 1 + z^6 = 0 \), hence \( z = e^{i(\pi + 2\pi k)/6} \). Thus the 6 poles are \( z_1 = e^{i\pi/6}, z_2 = e^{i\pi/2}, z_3 = e^{i5\pi/6}, z_4 = e^{i7\pi/6}, z_5 = e^{i0\pi/6}, z_6 = e^{i11\pi/6} \). 

-7
**Problem 4.b**

Thus we have

\[ f(z) = g(z) \cdot h(z) = \frac{1}{(z - e^{i\pi/6})(z - e^{i5\pi/6})(z - e^{i7\pi/6})(z - e^{i9\pi/6})(z - e^{i11\pi/6})} \]

also recall the properties of complex conjugates

**CASE 1:** \( \overline{uv} = (\overline{u})(\overline{v}) \)

**CASE 2:** \( \overline{u} + \overline{v} = \overline{u + v} \)

**CASE 3:** \( (\overline{u})^{-1} = \frac{1}{\overline{u}} \)

Hence

\[ h(z) = \frac{1}{(z - e^{-i\pi/6})(z - e^{-i9\pi/6})(z - e^{-i11\pi/6})} = g(z) \]

**Problem 5.a**

Notice the intersection of \(|z| = 1\) and \(|z - 1| = 1\) is at \(1/2 + i\sqrt{3}/2\) and at \(1/2 - i\sqrt{3}/2\). Then let

\[ T(z) = \frac{z - (1/2 - i\sqrt{3}/2)}{z - (1/2 - i\sqrt{3}/2)} \]

Then we map \(1/2 + i\sqrt{3}/2 \mapsto 0, 1/2 - i\sqrt{3}/2 \mapsto \infty\), and \(0 \mapsto 1\). Hence the \(|z - 1| = 1\) circle maps to the real axis. Now after some heavy computation, \(1 \mapsto -1/2 + i\sqrt{3}/2\), which is a line with \(\theta = 2\pi/3\). Hence the mapping

\[ w = \left( \frac{z - (1/2 - i\sqrt{3}/2)}{z - (1/2 - i\sqrt{3}/2)} \right)^{3/4} \]

maps \(L\) to the first quadrant. Now the heat equation is

\[ \frac{du}{dt} = \frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} \]

To find the steady state, we need \(u\) such that

\[ \frac{du}{dt} = 0 = \frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} \]

Hence we need \(u\) to be harmonic. Since the right boundary of \(L\) maps to the imaginary axis, and the left maps to the real axis, let’s find \(u\) such that it is 20 on the negative real axis, and 0 on the
positive real axis. By Poisson’s Formula in finding harmonic functions in the upper half plane, we have

\[
u(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(\zeta, 0)y}{y^2 + (x - \zeta)^2} d\zeta = \frac{1}{\pi} \int_{-\infty}^{0} \frac{20y}{y^2 + (x - \zeta)^2} d\zeta = 10 + \frac{20}{\pi} \arctan(y/x)\]

Then for

\[
w_2 = \left( \frac{z - (1/2 - i\sqrt{3}/2)}{z - (1/2 - i\sqrt{3}/2)}(-1/2 - i\sqrt{3}/2) \right)^{3/2}
\]

We set \(x = \text{Re}(w)\) and \(y = \text{Im}(w)\).

**Jan 1996 Linear Algebra**

**Problem 2.a**

SVD of an \(m \times n\) matrix \(A\) is when \(\exists\) orthonormal matrices \(U, V, \) and \(\Sigma\) such that

\[A = U\Sigma V^T\]

where \(U\) is \(m \times m\), \(\Sigma\) is \(m \times n\), and \(V\) is \(n \times n\). If \(\text{Rank}(A) = r\), the the first \(r\) column of \(V\) is the row space, and the last \(n - r\) columns of \(V\) is the null space of \(A\). The first \(r\) columns of \(U\) is the column space of \(A\) and the last \(m - r\) columns is the null space of \(A^T\). And

\[
\Sigma = \begin{pmatrix}
\sigma_1 & & \\
& \sigma_1 & \\
& & \ddots \\
& & & \sigma_r
\end{pmatrix}
\]

where \(\sigma_1 \geq \sigma_2 \geq \cdots \sigma_r > 0\).

**Problem 2.b**

We have

\[A = U\Sigma V^T \quad \text{and} \quad A^T = V\Sigma U^T\]

Then

\[AA^T = U\Sigma V^T V\Sigma U^T = U\Sigma^2 U^T\]

and

\[A^T A = V\Sigma U^T U\Sigma V^T = V\Sigma^2 V^T\]

since \(U^T = U^{-1}\) and \(V^T = V^{-1}\). Hence

\[(AA^T)U = U\Sigma^2 \quad \text{and} \quad (A^T A)V = V\Sigma^2\]

Hence \(\sigma_i^2\) are the eigenvalues of \(AA^T\) and \(A^T A\) and have the same multiplicity.
Problem 3.a

Recall the minmax theorem

\[ \lambda_i = \min_{S, \text{dim}(S) = i} \max_{x \in S, x \neq 0} \frac{x^T A x}{x^T x} \]

Let \( S = \{ x \in \mathbb{R}^4 : x_1 + x_2 = 0 \} \). Notice that \( \text{dim}(S) = 3 \). By the minmax theorem, notice

\[ 0 = \lambda_3 = \min_{S, \text{dim}(S) = 3} \max_{x \in S, x \neq 0} \frac{x^T A x}{x^T x} \max_{x \neq 0, x_1 + x_2 = 0} \frac{x^T A x}{x^T x} \leq \max_{x \neq 0} \frac{x^T A x}{x^T x} = \lambda_4 = 10 \]

Now notice that for \( x = (0, 0, 0, 1), x \in S \). Then

\[ \frac{x^T A x}{x^T x} = 1 > 0 \]

Hence we have a strict inequality for the lower bound. Now it suffices to show that the eigenvector \( v \) with eigenvalues 10 is not in \( S \). Assume that \( x = (x_1, -x_1, x_3, x_4) \) has an eigenvalue of 10. Then notice \( A x = 10 x \) implies \( x = 0 \). Hence we have a contradiction. Thus we have strict inequality.

Problem 3.b

We have

\[ u = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix} \quad v_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \]

Problem 3.c

We need to make a change a basis. Let

\[ M = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

Then \( M = M^{-1} \), and we will apply the change of basis

\[ MAM^{-1} = \begin{pmatrix} -1 & -6 & 5/\sqrt{2} & 11/\sqrt{2} \\ -6 & -9 & -11/\sqrt{2} & 5/\sqrt{2} \\ 5/\sqrt{2} & -11/\sqrt{2} & -4 & 4 \\ 11/\sqrt{2} & 5/\sqrt{2} & 4 & 1 \end{pmatrix} \]

Thus

\[ \max_{x_1 + x_2 = 0, x \neq 0} \frac{x^T A x}{x^T x} = \max_{x_1 = 0} \frac{x^T M A M^{-1} x}{x^T x} = \max_{x \in \mathbb{R}^3} \frac{x^T A' x}{x^T x} \]
where

\[ A' = \begin{pmatrix}
-9 & -11/\sqrt{2} & 5/\sqrt{2} \\
-11/\sqrt{2} & -4 & 4 \\
5/\sqrt{2} & 4 & 1
\end{pmatrix} \]

**Problem 4.a**

Notice indeed \( A \) and \( B \) have the same characteristic polynomial

\[
det(A - \lambda I) = det \begin{pmatrix}
-\lambda & 1 & 0 \\
0 & -\lambda & 1 \\
-2 & 3 & -\lambda
\end{pmatrix} = -\lambda(\lambda^2 - 3) - 2 = -\lambda^3 + 3\lambda - 2
\]

and

\[
det(B - \lambda I) = det \begin{pmatrix}
1 - \lambda & 0 & 0 \\
0 & 1 - \lambda & 0 \\
0 & 0 & -2 - \lambda
\end{pmatrix} = (1 - \lambda)^2(-2 - \lambda) = -\lambda^3 + 3\lambda - 2
\]

However if \( A \) and \( B \) were similar, then they would have the same: determinant, trace, size, nullity, characteristic polynomial, eigenvalues and vectors. So notice

\[
det A = 0 \neq -2 = det B
\]

**Problem 4.b**

Give that the characteristic polynomial of \( C \) is \((\lambda^2 - \sqrt{2}\lambda + 1)(\lambda - 1)\), we see that it’s eigenvalues are \( \lambda_1 = 1, \lambda_2 = \sqrt{2}/2 + i\sqrt{2}/2 \), and \( \lambda_3 = \sqrt{2}/2 - i\sqrt{2}/2 \). Let \( Q \) be defined as

\[
Q = \begin{pmatrix}
1 & 0 & 0 \\
0 & \sqrt{2}/2 + i\sqrt{2}/2 & 0 \\
0 & 0 & \sqrt{2}/2 - i\sqrt{2}/2
\end{pmatrix}
\]

Then notice \( QQ^T = Q(Q)^T = I = (Q)^TQ = Q^TQ \). Hence \( Q \) is indeed orthogonal (in the complex way). Also since \( C \) has 3 distinct eigenvalues, it has 3 independent eigenvectors \( x_1, x_2, \) and \( x_3 \). Thus we define

\[
M = \begin{pmatrix}
| & | & | \\
x_1 & x_2 & x_3
\end{pmatrix}
\]

\( M \) is invertible and \( CM = MQ \). Hence \( C \) and \( Q \) are similar.
Problem 5.a

So we have

$$\begin{pmatrix}
2 & 1 & 0 & \cdots & 0 \\
1 & 4 & 2 & 0 & \\
2 & 6 & & & \\
\ddots & n-1 & & & \\
n-1 & n-1 & & 2n \\
\end{pmatrix} x = 0$$

Now I claim that the matrix (we will call $A$) is invertible. Notice that the determinant $D_n$ is

$$D_n = 2nD_{n-1} - (n-1)^2D_{n-2} = \prod \lambda_i > 0$$

for all $n$ and $D_1 = 2, D_2 = 7, D_3 = 36$. Hence $D_n > 0$.

Problem 5.b

Recall that for symmetric matrices TFAE: 1) all $n$ eigenvalues are greater than 0, 2) all upper left determinant is greater than 0, 3) all $n$ pivots are greater than 0, 4) $x^T Ax > 0$ for $x \neq 0$. By part (a), 2 is satisfies which implies 1 is satisfied.

Sept 1996 Advanced Calculus

Problem 1.a

We know that the product

$$\prod_{n=1}^{\infty} (1 + a_n)$$

converges absolutely if and only if

$$\sum_{n=1}^{\infty} a_n$$

converges absolutely. Thus assume that $\lim_{n} f(p,n)$ converges. Then

$$\sum_{k=1}^{\infty} \frac{p}{k} < \infty$$

which is a contradiction. Hence $f(p,n) \to \infty$ as $n \to \infty$. 

-12
Problem 1.b

Recall that
\[ \sum_{k=1}^{n} \frac{1}{k} \geq \int_{1}^{n+1} \frac{1}{x} \, dx = \log(n + 1) \]

Hence
\[ \lim_{n \to \infty} \left| \log n - \sum_{k=1}^{n} \frac{1}{k} \right| = \lim_{n \to \infty} |\log(n) - \log(n + 1)| = \lim_{n \to \infty} |\log(n/(n + 1))| \to 0 \]

Hence it converges and the limit exists.

Problem 1.c

We know that \( \lim_{n} \frac{f(p,n)}{n^p} \) exists if and only if
\[ \lim_{n \to \infty} \log \left( \frac{f(p,n)}{n^p} \right) \]

notice
\[ = \lim_{n \to \infty} \left( \sum_{k=1}^{\infty} \log(1 + p/k) - \log n^p \right) = \lim_{n \to \infty} \sum_{k=1}^{\infty} \sum_{s=1}^{\infty} \frac{(p/k)^s}{s} (-1)^{s+1} - \log n^p \]
\[ = \left( \sum_{k=1}^{\infty} \frac{p}{k} - \log n^p \right) + \sum_{k=2}^{\infty} \sum_{s=1}^{\infty} \frac{p^s}{sk^s} (-1)^{s+1} \]

by part (b) it converges.

Problem 2.a

For \( f(x) = xe^x \), we have \( f(0) = 0 \) and \( \lim_{x \to \infty} f(x) \to \infty \). Also \( f \) is continuous on \( x > 0 \). Therefore for any positive \( t \), \( \exists x_*(t) \) such that \( t = x_*(t)e^{x_*(t)} \).

Problem 2.b

\[ \lim_{t \to \infty} \frac{x_*(t)}{\log t} = \lim_{x \to \infty} \frac{x}{\log(xe^x)} = \lim_{x \to \infty} \frac{x}{x + \log x} = 1 \]

Problem 4

Now if \( \nabla \times E = 0 \) implies that \( \exists \) a potential function, that is gradient to it. I.e. \( \exists \phi \) such that
\[ E = \nabla \phi \]

So
\[ \nabla \cdot (\phi J) = \nabla \phi \cdot J + \phi \nabla \cdot J = \nabla \phi \cdot J = E \cdot J \]
since $\nabla \cdot J = 0$. So

$$\int_{\Omega} E \cdot J \, dx = \int_{\Omega} \nabla \cdot (\phi J) \, dx = \int_{\partial(\Omega)} \phi J \cdot dS$$

Since $E = 0$ on the neighborhood of the boundary, then $\phi$ is constant on the boundary. So there is a potential $\psi$ such that $\psi = 0$ on the boundary. Hence

$$\int_{\Omega} E \cdot J \, dx = \int_{\partial(\Omega)} \psi J \cdot dS = 0$$

Sept 1996 Complex Variables

Problem 1

Here we fall into two cases. **CASE 1:** let $a$ be purely imaginary, $a = iv$ where $v \in \mathbb{R}$. Then notice

$$\int_{-\infty}^{\infty} e^{-t^2/2-v^2} dt = \int_{-\infty}^{\infty} e^{-(t+v)^2/2+v^2} dt = e^v \int_{-\infty}^{\infty} e^{-(t+iv)^2/2} dt = e^v \sqrt{2\pi}$$

**CASE 2:** Now assume that $a = u + iv$ where $v \neq 0$. Then let $f(z) = e^{-z^2/2}$ and $\gamma$ be the contour from $-R$ to $R$, to $R-ia$, to $-R-ia$, to $-R$. Then by the Residue Theorem, we have

$$\int_{\gamma} f(z) \, dz = \int_{-\infty}^{\infty} e^{-t^2/2} dt + \int_{0}^{1} e^{-(R-iat)^2/2} iadt + \int_{1}^{0} e^{-(R-iat)^2/2} iadt + \int_{\infty}^{-\infty} e^{-(t-ia)^2/2} dt = 0$$

Now notice

$$\left| \int_{0}^{1} e^{-(R-iat)^2/2} iadt \right| \leq \sqrt{u^2 + v^2} e^{-R^2/2 - Rv + u^2/2} \to 0$$

as $R \to \infty$. Likewise we have

$$\left| \int_{1}^{0} e^{-(R-iat)^2/2} iadt \right| \to 0$$

as $R \to \infty$. Hence as $R \to \infty$ we have

$$\int_{-\infty}^{\infty} e^{-t^2/2} dt + \int_{-\infty}^{\infty} e^{-(t-ia)^2/2} dt = 0$$

Hence

$$\int_{-\infty}^{\infty} e^{-t^2/2-2\pi i w} dt = \sqrt{2\pi} e^{-2\pi^2 w^2}$$
Problem 2
By the Poisson Integral formula, we have

\[ \phi(x, y) = \frac{1}{\pi} \int_{-\infty}^{0} \frac{-y}{y^2 + (x-w)^2} \, dw + \frac{1}{\pi} \int_{0}^{\infty} \frac{y}{y^2 + (x-w)^2} \, dw \]

By letting \( u = (x-w)/y \) and \( du = -dw/y \), we have

\[ \frac{1}{\pi} \left[ \arctan \left( \frac{x-w}{y} \right) \right]_{0}^{\infty} - \frac{1}{\pi} \left[ \arctan \left( \frac{x-w}{y} \right) \right]_{-\infty}^{0} = \frac{2}{\pi} \arctan \left( \frac{x}{y} \right) \]

which is bounded. Notice if \( \phi \) did not have to be bounded, we have

\[ \phi(x, y) = \frac{2}{\pi} \arctan \left( \frac{x}{y} \right) + Cy \]

for constant \( C \).

Problem 3
First we will map \( D = \{ z : |z| < 1, 0 < \arg(z) < \pi/2 \} \) to the upper-half unit circle by \( w_1 = z^2 \).

Then we will map the upper-half unit circle to the first quadrant by

\[ w_2 = i \left( \frac{1 - z^2}{1 + z^2} \right) \]

Then we will map that the the UHP by

\[ w_3 = -\left( \frac{1 - z^2}{1 + z^2} \right)^2 \]

Then we map the UHP to the interior of the unit circle by

\[ w_f = -\left( \frac{1 - z^2}{1 + z^2} \right)^2 - i \left( \frac{1 - z^2}{1 + z^2} \right)^2 + i \]

Problem 5
Let \( f \) be an entire function. Then assume \( \exists \) a fixed \( \epsilon > 0 \) and \( a \in \mathbb{C} \) such that for all \( z \in \mathbb{C} \) we have

\[ |f(z) - a| \geq \epsilon \]

Now let \( g(z) = 1/(f(z) - a) \). Then \( g \) is an entire function such that

\[ |g(z)| = \frac{1}{|f(z) - a|} \leq \frac{1}{\epsilon} \]

Thus \( g \) is an entire bounded function. By Liouville’s Theorem, \( g \) must be constant
\[ g(z) = c = \frac{1}{f(z) - a} \]

which implies \( f \) is constant. But since \( f \) assumes 1 and 0, we have a contradiction.

**Sept 1996 Linear Algebra**

**Problem 1**

If \( v_1 \) and \( v_2 \) are the eigenvectors of \( A \) with eigenvalues \( \lambda_1 \) and \( \lambda_2 \), then \( v_1 \) and \( v_2 \) are eigenvectors of \( A^2 \) with eigenvalues \( \lambda_1^2 \) and \( \lambda_2^2 \). So by applying the quadratic equation, we have

\[ t = -1 \pm i\sqrt{2} \]

Thus we have \( \lambda_1 = -1 + i\sqrt{2} \) and \( \lambda_2 = -1 - i\sqrt{2} \). So this implies \( \lambda_1^2 = -1 - 2i\sqrt{2} \) and \( \lambda_2^2 = -1 + 2i\sqrt{2} \). Therefore the characteristic equation for \( A^2 \) is

\[ (t - (-1 - 2i\sqrt{2}))(t - (-1 + 2i\sqrt{2})) = t^2 + 2t + 9 \]

**Problem 2.1**

notice

\[ \det \begin{pmatrix} \alpha + 1 - \lambda & \alpha \\ \alpha - 1 & \alpha - \lambda \end{pmatrix} = \lambda^2 + (-2\alpha - 1)\lambda + 2\alpha \]

Thus \( \lambda_1 = 2\alpha \) and \( \lambda_2 = 1 \). Hence

\[ \text{Tr}(B) = \sum_{i=0}^{m} (2\alpha)^i + 1 = m + \frac{1 - (2\alpha)^{m+1}}{1 - 2\alpha} \]

**Problem 2.2**

Now if \( \alpha \neq 1/2 \), then we can diagonalize \( A \), and notice

\[ A^i = M \left( \begin{array}{c} 1 \\ (2\alpha)^i \end{array} \right) M^{-1} \neq 0 \]

as \( i \to \infty \). Thus the sum does not converge. Now if \( \alpha = 1/2 \), then

\[ A^n = \frac{1}{2} \left( \begin{array}{cc} n + 2 & n \\ -n & 2 - n \end{array} \right) \neq 0 \]

as \( n \to \infty \). Thus the summation does not converge for any \( \alpha \).
Problem 3.1

Let \( x \in \text{Im}(A) \). Then \( \exists y \) such that \( Ay = x \). Thus notice

\[
Ax = A(Ay) = A^2y = 0y = 0
\]

hence \( x \in \ker(A) \Rightarrow \text{Im}(A) \subset \ker(A) \).

Problem 4

Notice by Gaussian elimination we have

\[
\begin{array}{ccc|c}
2 & s + 1 & 1 & -a \\
s + 4 & 2 & 2 & 3 \\
2s & 2s & s & 4 \\
\hline
2 & s + 1 & 1 & -a \\
0 & -(s + 4)(s + 1)/2 & -(s + 4)/2 + 2 & 3 + a(s + 4)/2 \\
0 & -s^2 + s & 0 & 4 + as \\
\end{array}
\]

Hence in order to have an infinite number of solutions, we must have \( s = 1 \) and \( a = -4 \). Hence by back substitution we have

\[
s = (3/5, 7/5, 0) + t(-3/5, 1/10, 1)
\]

Problem 5.1

If the entries on the diagonal of \( L \) are positive and real, then all pivots are positive, and thus it is positive definite.

Problem 5.2

\[
A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}
\]

Problem 5.3

We want to show that the Cholesky Decomposition is unique. Notice

\[
A = LL^T = BB^T
\]

Since \( L \) and \( B \) have all positive entries, it is invertible. Now

\[
L^T(B^T)^{-1} = L^{-1}B \Rightarrow L^T(B^{-1})^T = L^{-1}B \Rightarrow (B^{-1}L)^T = (B^{-1}L)^{-1}
\]

Now since \( L \) and \( B \) are lower triangular, \( L^{-1} \) and \( B^{-1} \) is lower triangular, And thus \( B^{-1}L \) and \( (B^{-1}L)^{-1} \) is lower triangular Now since

\[
(B^{-1}L)^T = (B^{-1}L)^{-1}
\]
we have

\[(B^{-1}L)^T = (B^{-1}L)^{-1} = D\]

for some diagonal matrix \(D\). Hence

\[L^{-1}BL^T(B^{-1})^T = D^2 = (B^{-1}L)(B^{-1}L)^{-1} = I\]

since \(B^{-1}L\) is symmetric. Thus \(D\) has \(\pm 1\) on its diagonal. But since all entries are positive, we have \(D = I\). Therefore \(B = L\), and the factorization is unique.

Jan 1997 Advanced Calculus

Problem 1

Let \(x = 1/s\), then \(dx = (-1/s^2)ds\) and we have

\[
\lim_{\epsilon \to 0} \int_{1/\epsilon}^{1} s^\alpha \sin(1/s)ds = \lim_{\epsilon \to 0} \int_{1/\epsilon}^{1/\epsilon} -\sin x / x^{\alpha-2} dx = \lim_{\epsilon \to 0} \int_{1}^{1/\epsilon} \sin x / x^{\alpha-2} dx = \int_{1}^{\infty} \sin x / x^{\alpha-2} dx
\]

Now for \(\alpha > 3\) then clearly the integral converges absolutely, which implies that it converges. For \(\alpha \leq 2\), notice

\[
\lim_{x \to \infty} \frac{\sin x}{x^{\alpha-2}} \neq 0
\]

hence the integral does not exist. Finally for \(2 < \alpha \leq 3\), we have

\[
= \int_{1}^{\infty} \frac{\sin x}{x^{p}} dx
\]

where \(0 < p \leq 1\). Using integration by parts, we have \(u = 1/x^p\), \(du = -px^{-p-1}\), \(v = -\cos x\) and \(dv = \sin x dx\). Thus

\[
= -\frac{\cos x}{x^p} \bigg|_{1}^{\infty} - p \int_{1}^{\infty} \frac{\cos x}{x^{p+1}} dx < \infty
\]

Hence the integral exists for \(\alpha > 2\).

Problem 2

Let \(u = a \cdot r = a_1x + a_2y + a_3z\), \(v = b \cdot r = b_1x + b_2y + b_3z\), and \(w = c \cdot r = c_1x + c_2y + c_3z\). Then

\[
\begin{align*}
    du &= a_1dx + a_2dy + a_3dz \\
    dv &= b_1dx + b_2dy + b_3dz \\
    dw &= c_1dx + c_2dy + c_3dz
\end{align*}
\]

Hence
\[ dudvdw = \det \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \, dx dy dz \]

Hence

\[
\int \int \int_E (a \cdots r)(b \cdot r)(c \cdot r) \, dx dy dz &= \frac{1}{|(a \cdot b \times c)|} \int_0^\gamma \int_0^\beta \int_0^\alpha uvwdudvdw = \frac{(\alpha\beta\gamma)^2}{8|(a \cdot b \times c)|} \\
\]

**Problem 3.a**

We have \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) for all real \( x \). \( f(0) = 1 \Rightarrow a_0 = 1 \). Notice

\[
0 = x^2 f''(x) + xf'(x) + x^2 f(x) = (2a_2 x^2 + 6a_3 x^3 + 12a_4 x^4 + 20a_5 x^5 + 30a_6 x^6 + \cdots) \\
+ (a_{1}x + 2a_2 x^2 + 3a_3 x^3 + 4a_4 x^4 + 5a_5 x^5 + 6a_6 x^6 + \cdots) + (x^2 + a_1 x^3 + a_2 x^4 + a_3 x^5 + \cdots)
\]

\[
= a_1 x + (1 + 2a_2 + 2a_2)x^2 + (a_1 + 3a_3 + 6a_6)x^3 + \cdots
\]

Hence \( a_1 = 0 \), and \( a_2 = -1/4 \).

**Problem 3.b**

By above we can see that

\[
a_{n-2} + na_n + n(n-1)a_n = 0
\]

for \( n \) even , and \( a_n = 0 \) for \( n \) odd. Thus

\[
a_n = \begin{cases} 
0 & \text{n is odd} \\
-\frac{a_{n-2}}{n^2} & \text{n is even}
\end{cases}
\]

which implies

\[
a_n = \begin{cases} 
0 & \text{n is odd} \\
\frac{(i)^n}{n^2(n-2)^2 \cdots 2^2} & \text{n is even}
\end{cases}
\]

where \( a_0 = 1 \)

**Problem 3.c**

By letting

\[
g(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 \cdots
\]

and solving the equation given and applying part (b), we can see that it equals zero.
**Problem 4.a**

Notice for $x > 0$, we have by L’Hospital’s rule,

$$\lim_{t \to 0} \frac{e^{-xt} - e^{-t}}{t} = \frac{-xe^{-xt} + e^{-t}}{1} = -x + 1$$

Thus it suffices to show that for

$$\phi(x) = \int_0^1 \frac{e^{-xt} - e^{-t}}{t} \, dt + \int_1^\infty \frac{e^{-xt} - e^{-t}}{t} \, dt$$

that

$$\int_1^\infty \frac{e^{-xt} - e^{-t}}{t} \, dt < \infty$$

Notice by using integration by parts, we have $u = 1/t$, $du = -1/t^2$, $v = e^{-xt}/x + e^{-t}$ and $dv = e^{-xt} - e^{-t}$ implies

$$= \frac{1}{x} \left( \frac{e^{-xt}}{x} + e^{-t} \right) \bigg|_1^\infty + \int_1^\infty \frac{e^{-xt}}{x} + \frac{e^{-t}}{t^2} \, dt$$

and

$$\int_1^\infty \frac{e^{-xt}}{x} + \frac{e^{-t}}{t^2} \, dt$$

converges absolutely and so it converges.

**Problem 4.b**

Notice

$$\phi'(x) = \int_0^\infty -e^{-xt} \, dt = -\frac{1}{x}$$

**Problem 4.c**

Finally by part (b), we potentially have

$$\phi(x) = -\log x + C$$

Notice

$$\lim_{R \to \infty} \int_R^\infty \left| \frac{e^{-xt} - e^{-t}}{t} \right| \, dt \leq \lim_{R \to \infty} \frac{1}{R} \int_R^\infty e^{-xt} + e^{-t} \, dt = \lim_{R \to \infty} \frac{1}{R} \left( \frac{e^{-Rx}}{x} + e^{-R} \right) \to 0$$

Hence

$$\lim_{R \to \infty} \int_R^\infty \frac{e^{-xt} - e^{-t}}{t} \, dt = 0$$
uniformly for $x \in (0, \infty)$. Thus
\[
\phi(1) = \int_0^\infty 0\,dt = 0 = -\log(1) + C
\]
which implies $C = 0$. Therefore $\phi(x) = -\log x$.

**Problem 5**

Notice we have
\[
\lim_{t \to 0} \int_{-t}^t g(x) \frac{1}{t} \left(1 - \frac{x^2}{t^2}\right) \,dx = \lim_{t \to 0} \frac{1}{t^3} \int_{-t}^t g(x)(t^2 - x^2) \,dx = \lim_{t \to 0} \frac{1}{t^3} \sum_{k=1}^n \frac{2t}{n} g(-t + k\frac{2t}{n})(t^2 - (-t + k\frac{2t}{n})^2)
\]
\[
= \lim_{t \to 0} \frac{1}{t^3} \lim_{n \to \infty} \sum_{k=1}^n \frac{2t}{n} g(-t + 2kt/n)(k4t^2/n - k^2t^2/n^2) = \lim_{n \to \infty} \sum_{k=1}^n \frac{2}{n} g(0) \left(\frac{4k}{n} - \frac{k^2}{n^2}\right)
\]
\[
= 8g(0) \lim_{n \to \infty} \sum_{k=1}^n \left(\frac{k}{n^2} - \frac{k^2}{n^3}\right) = 8g(0) \lim_{n \to \infty} \frac{1}{n^3} \sum_{k=1}^n k_n - k^2
\]
\[
= 8g(0) \frac{1}{n^3} \left(\frac{n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6}\right) = 8g(0) \lim_{n \to \infty} \frac{1}{n^3} \left(\frac{n^3 - n}{6}\right) = \frac{4}{3}g(0)
\]

**Jan 1997 Complex Variables**

**Problem 1**

Let $\gamma$ be the semi-circle in the upper half plane with radius $R$. Then let $f(z) = \frac{e^{iz}}{(z+ia)^2(z-ia)^2}$. Then by the Residue Theorem, we have
\[
\int_{\gamma} f(z) \,dz = \lim_{z \to -ia} 2\pi i \left(\frac{ie^{iz}(z + ia)^2 - 2(z + ia)e^{iz}}{(z + ia)^4}\right) = \frac{e^{-a\pi}}{4a^3}(2a + 2)
\]
Now notice on the outer circle $z = Re^{i\theta}$, we have
\[
\left|\int_{C_R} f(z) \,dz\right| \leq \frac{\pi Re^{-r \sin \theta}}{(R^2 - a)^2} \to 0
\]
as $R \to \infty$. Thus as $R \to \infty$
\[
\int_{-\infty}^{\infty} f(z) \,dz = \frac{e^{-a\pi}}{4a^3}(2a + 2)
\]
Problem 2.a
Let $f(z) = z^n e^A - e^z$. Then on $|z| = 1$ we have

$$|z^n e^A - e^z - z^n e^A| = |e^z| = e|z| = e < e^A = |z^n e^A|$$

Hence $f(z)$ has $n$ roots inside $|z| < 1$.

Problem 2.b
To see that the root is real and positive, it suffices to show $\exists x \in (0, 1)$ such that

$$xe^A = e^x$$

Let $g(x) = xe^A - e^x$. Then $g$ is clearly continuous and $g(0) = -1$ and $g(1) = e^A - e > 0$. Thus by the intermediate value theorem, $\exists x_0 \in (0, 1)$ such that $g(x_0) = 0$. Hence $x_0$ is the real positive root such that

$$x_0 e^A - x_0 = 1$$

Problem 3.a
For $|z| > 1$, notice $\exists M$ such that

$$\sum_{n=1}^{\infty} \left| \frac{z^n}{(1 + z^n)(1 + z^{n+1})} \right| \leq \sum_{n=1}^{\infty} \left( \frac{|z|^n}{|z|^n - 1} \right) \leq \sum_{n=1}^{\infty} \left\{ \frac{|z|^n}{|z|^n - 1} \right\}$$

and so

$$\leq \sum_{n=1}^{\infty} M|z|^n \leq M \sum_{n=1}^{\infty} \frac{|z|^n}{1 - |z|^n} = \frac{M|z|}{|z| - 1}$$

Since $|z| > 1$. Thus it converges absolutely. Now for $|z| < 1$, notice

$$\sum_{n=1}^{\infty} \left| \frac{z^n}{(1 + z^n)(1 + z^{n+1})} \right| \leq \sum_{n=1}^{\infty} \left( \frac{|z|^n}{1 - |z|^n} \right)$$

Now $\exists M$ such that

$$\leq \sum_{n=1}^{M} \frac{|z|^n}{(1 - |z|^n)(1 - |z|^n+1)} + \sum_{n=M+1}^{\infty} \frac{|z|^n}{(1/2)(1/2)} < \infty$$
Problem 3.b

By the hint we have

\[
\sum_{n=1}^{\infty} \frac{z^n}{(1 + z^n)(1 + z^{n+1})} = \sum_{n=1}^{\infty} \frac{1}{z + 1} \left[ 1 + \frac{1}{1 + z^n} - \frac{1}{1 + z^{n+1}} \right] = \frac{1}{z - 1} \sum_{n=1}^{\infty} \frac{1}{1 + z^n} - \frac{1}{1 + z^{n+1}}
\]

\[
= \frac{1}{z - 1} \left( \frac{1}{1 + z} + \frac{1}{1 + z^2} + \frac{1}{1 + z^3} + \cdots - \frac{1}{1 + z} - \frac{1}{1 + z^2} - \cdots \right) = \frac{1}{z^2 - 1}
\]

Problem 3.c

Same as above but must be careful.

Problem 4.a

Notice that

\[
w = z + \frac{1}{z} = (x + iy) + \frac{1}{x + iy} = \left( x + \frac{x}{x^2 + y^2} \right) + i \left( y - \frac{y}{x^2 + y^2} \right)
\]

So on the \(x^2 + y^2 = 1\) border, we have \(w = 2x \in [-2, 2]\). Also for \(y = 0\), we have \(w = x + 1/x \in (-\infty, -2) \cup (2, \infty)\). Since \(i/2 \rightarrow -3/2i\), \(w\) maps the region to the LHP.

Problem 4.b

By the Poisson integral formula, we have

\[
\phi(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{-yG(w)dw}{y^2 + (x - w)^2} = \int_{-2}^{2} \frac{-y}{y^2 + (x - w)^2} dw = \int_{-2}^{2} \frac{-(1/y)}{1 + \left( \frac{x-w}{y} \right)^2} dw
\]

now by \(u\)-substitution, we have \(u = (x - w)/y\) and \(du - dw/y\). Thus

\[
= \frac{1}{\pi} \int \frac{1}{1 + u^2} du = \frac{1}{\pi} \arctan \left( \frac{x - w}{y} \right) \bigg|_{-2}^{2} = \frac{1}{\pi} \arctan \left( \frac{x - 2}{y} \right) - \frac{1}{\pi} \arctan \left( \frac{x + 2}{y} \right)
\]

So since

\[
w = \left( x + \frac{x}{x^2 + y^2} \right) + i \left( y - \frac{y}{x^2 + y^2} \right)
\]

we have

\[
\phi(x, y) = \frac{1}{\pi} \arctan \left( \frac{x + \frac{x}{x^2 + y^2} - 2}{y - \frac{y}{x^2 + y^2}} \right) - \frac{1}{\pi} \arctan \left( \frac{x + \frac{x}{x^2 + y^2} + 2}{y - \frac{y}{x^2 + y^2}} \right)
\]
Problem 5.a

We know that by the max/min principle, \( f \) is analytic and has no zeros inside \(|z| \leq 1\) implies that \( g(z) = 1/f(z) \) is analytic and again by the min/max principle, \(|g(z)| = |1/f(z)| \leq 1\). This implies that \(|f(z)| = 1\) in \(|z| \leq 1\). Therefore recall that if \(|f|, Re(f), \text{ or } Im(f)\) is constant, then \( f \) is constant. Notice if \(|f| = 1\), then \( f = 1/\overline{f} \) and so \( \overline{f} \) is analytic. This implies \( g = f + \overline{f} \) is analytic and \( Im(g) = 0 \). Then by Cauchy Riemann equations, we can see that \( g \) is constant and hence \( f \) is constant. So \( f = e^{i\beta}\).

Problem 5.b

Then let \( g(z) = f(z)/z^m \). Then \( g \) is analytic in \(|z| < 1\) and we can apply part (a) to see that \( f(z) = z^m e^{i\beta} \).

Jan 1997 Linear Algebra

Problem 1

The operator for \( xd/dx \) in matrix form is the \( (n+1) \times (n+1) \) matrix

\[
T = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 2 & 0 & \cdots & 0 \\
0 & 0 & 3 & \cdots & 0 \\
\vdots & \vdots & 0 & \ddots & 0 \\
0 & 0 & \cdots & 0 & n+1
\end{pmatrix}
\]

And so the eigenvalues are \( \lambda \in \{1, 2, \ldots, n, n+1\} \) with eigenvectors \( e_i \) for \( i \leq 1 \) to \( n+1 \). Now for the operator \( d/dx \) we have the matrix form

\[
D = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 \\
0 & 0 & \cdots & 0 & n \\
0 & 0 & \cdots & 0 & 0
\end{pmatrix}
\]

Thus the eigenvalues are all 0 with eigenvector \( \alpha e_1 \) where \( \alpha \in \mathbb{R} \).

Problem 2.a

Let \( P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \) be the projection on the \( x \) axis and \( P_2 = (1/2) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \) be the projection on the \( y = x \) line. Then

\[
P_1P_2 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = A
\]

But \( A^2 = 1/2A \neq A \). Hence it is not a projection.
Problem 2.b

For $P_2P_1$ to be a projection, we must have $(P_2P_1)^2 = P_2P_1P_1P_1 = P_2P_1$. This implies we must have either $P_1$ and/or $P_2 = I$, or $M_1 \perp M_2$, or $P_1 = P_2$. Then we would have $P_2P_1$ be a projection.

Problem 3

We will apply Gram Schmidt on $e_1, e_2, e_3$. Then

$$u_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Now

$$u_2 = e_2 - (e_2, u_1)u_1$$

Since $(e_2, u_1) = 0$, after normalizing it, we have

$$u_2 = \frac{1}{2} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Finally

$$u_3 = e_3 - (e_3, u_1)u_1 - (e_3, u_2)u_2$$

and since $(e_3, u_1) = 1/\sqrt{2}$ and $(e_3, u_2) = 0$, we have

$$u_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{2}} \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1/2 \\ \phantom{-}0 \\ \phantom{-}1 \end{pmatrix}$$

after normalizing it, we have

$$u_3 = \sqrt{\frac{2}{3}} \begin{pmatrix} -1/2 \\ \phantom{-}0 \\ \phantom{-}1 \end{pmatrix}$$

Problem 4

We are given that

$$A \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \quad A \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1/2 \\ \phantom{-}1/2 \end{pmatrix}, \quad A \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ \phantom{-}1 \\ \phantom{-}1 \end{pmatrix}$$

So after some manipulation, we have

$$A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{5} \\ \frac{1}{2} \\ \frac{1}{6} \end{pmatrix}, \quad A \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{6} \\ \frac{1}{2} \\ \frac{1}{3} \end{pmatrix}, \quad A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -5/6 \\ 0 \\ \frac{1}{6} \end{pmatrix}$$

-25
So

\[ A = \begin{pmatrix} \frac{1}{3} & -\frac{5}{6} & \frac{1}{6} \\ \frac{-1}{2} & 0 & \frac{1}{2} \\ \frac{-1}{6} & 1 & \frac{-1}{6} \end{pmatrix} \]

Since \( v_1, v_2, v_3 \) are linearly independent, we have \( S(v_1, v_2, v_3) = \mathbb{R}^3 \). So for any \( v = a_1v_1 + a_2v_2 + a_3v_3 \) we have

\[ A^n v = a_1 \left( \frac{1}{2} \right)^n v_2 + a_2 \left( \frac{1}{2} \right)^n + a_3v_3 \]

so as \( n \to \infty \), we have

\[ A^n v = \begin{pmatrix} -a_3 \\ a_3 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \]

**Problem 5.a**

**TRUE:** The fact that \( Ax = b \) has no solutions doesn’t really matter. Recall that

\[ R(A) + n(A) = n \]

Notice \( R(A) \leq m < n \) since \( A \) is an \( m \times n \) matrix with \( m \leq n \). Thus \( n(A) = n - R(A) > 0 \). Thus \( Ax = 0 \) has infinitely many solutions.

**Problem 5.b**

**FALSE:** Notice \( \left( \begin{array}{c} 1 \\ 2 \end{array} \right) x = \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \), which has no solutions. However

\[ \left( \begin{array}{c} 1 \\ 2 \end{array} \right) x = \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \]

has only one solution \( (x = 0) \).

**Problem 5.c**

**FALSE:** Notice that \( b \in \mathbb{R}^m \). So if \( A \) has \( m \) linearly independent column vectors (which is possible since \( m < n \)) say \( (v_1, v_2, ..., v_m) \). Then we have

\[ S(v_1, ..., v_m) = \mathbb{R}^m \]

So this implies \( b \in S(v_1, ..., v_m) \) and so \( \exists \) a solution for \( Ax = b \).

**Problem 5.d**

**TRUE:** If we are given that \( Ax = 0 \) has infinitely many solutions, this implies that \( n(A) > 0 \). Now we have the equation \( R(A) + n(A) = n \), and we also know that \( R(A) < n < m \), this implies \( A \) has at most \( i < n < m \) linearly independent column vectors \( v_1, ..., v_i \). Hence since \( i < m \), \( S(v_1, ..., v_i) \subseteq \mathbb{R}^i \). Thus \( \exists b \in \mathbb{R}^m \), such that \( b \notin S(v_1, ..., v_i) \).
Problem 1

Since $E(u) \leq E(u + \phi)$, Then pick $\phi = \epsilon \phi$. Then let

$$g(\epsilon) = E(u + \epsilon \phi)$$

Then $g'(0) = 0$, since $E$ is smooth and it minimizes at $\epsilon = 0$. Then

$$g(\epsilon) = \int \int \left( \frac{dv}{dx} + \epsilon \frac{d\phi}{dx} \right)^2 + \left( \frac{dv}{dy} + \epsilon \frac{d\phi}{dy} \right)^2 dx dy$$

$$= \int \int \left( \frac{dv}{dx} \right)^2 + 2\epsilon \frac{dv}{dx} \frac{d\phi}{dx} + \epsilon^2 \left( \frac{d\phi}{dx} \right)^2 + \left( \frac{dv}{dy} \right)^2 + 2\epsilon \frac{dv}{dy} \frac{d\phi}{dy} + \epsilon^2 \left( \frac{d\phi}{dy} \right)^2 dx dy$$

Now all are smooth and defined on a compact set. So when we differentiate with respect to $\epsilon$,

$$g'(0) = 2 \int \int \left( \frac{du}{dx} \right) \left( \frac{d\phi}{dx} \right) + \left( \frac{du}{dy} \right) \left( \frac{d\phi}{dy} \right) dx dy = 2 \int \int \nabla u \cdot \nabla \phi dx dy = 0$$

Now just like integration by part, notice by the divergence theorem

$$\int_{\delta D} \phi \nabla u \cdot dn = \int \int_D \nabla \cdot (\phi \nabla u) dx dy = \int \int_D (\nabla \phi \cdot \nabla u + \phi \nabla^2 u) dx dy$$

which implies

$$\int \int_D \phi \nabla u \cdot dn - \int \int_D \phi \nabla^2 u dx dy = \int \int_D (\nabla \phi \cdot \nabla u) dx dy$$

Since $\phi = 0$ on $\delta(D)$, we have

$$\int \int_D \phi \Delta u dx dy = 0$$

Since this is true for arbitrary $\phi$ implies $\Delta u = 0$.

Problem 2

Recall that $\cos(mx + nx) = \cos(mx) \cos(nx) - \sin(mx) \sin(nx)$ and $\cos(mx - nx) = \cos(mx) \cos(nx) + \sin(mx) \sin(nx)$. Thus

$$\frac{1}{2} (\cos(mx - nx) + \cos(mx + nx) = \cos(mx) \cos(nx)$$

which implies

$$\int \cos(mx) \cos(nx) dx = \frac{\sin(mx + nx)}{2(m + n)} + \frac{\sin(mx - nx)}{2(m - n)} + C$$

Also recall that $\cos(2x) = (1/2)(1 + \cos(2x))$. Now
\[
\int_{-1}^{1} \cos^4(x) \cos(nx) \, dx = \int_{-1}^{1} \frac{1}{4}(1 + \cos(2x))^2 \cos(nx) \, dx = \int_{-1}^{1} \left( \frac{1}{4} + \frac{1}{2} \cos(2x) + \frac{1}{4} \cos^2(2x) \right) \cos(nx) \, dx
\]

\[
= \int_{-1}^{1} \frac{1}{4} \cos(nx) \, dx + \int_{-1}^{1} \frac{1}{2} \cos(2x) \cos(nx) \, dx + \frac{1}{4} \int_{-1}^{1} \cos^2(2x) \cos(nx) \, dx
\]

and we can break down

\[
\frac{1}{4} \int_{-1}^{1} \cos^2(2x) \cos(nx) \, dx = \frac{1}{4} \int_{-1}^{1} \frac{1}{4}(1 + \cos(4x)) \cos(nx) \, dx = \frac{1}{16} \int_{-1}^{1} \cos(nx) \, dx + \frac{1}{16} \int_{-1}^{1} \cos(4x) \cos(nx) \, dx
\]

By what we have proven above, all integrals will converge to 0 as \( n \to \infty \). Hence

\[
\lim_{n \to \infty} \int_{-1}^{1} \cos^4(x) \cos(nx) \, dx = 0
\]

**Problem 4**

Notice that \( (1 + u^2/n)^n / e^{u^2} \). Thus by applying the alternating series theorem, we series converges pointwise. However it does not converge absolutely since

\[
\sum \frac{1}{\sqrt{n}(1 + u^2/n)^n} \geq \sum \frac{1}{\sqrt{n}e^{u^2}} \to \infty
\]

Now to show that it converges uniformly we will apply Abel’s Test: let

\[
\sum \frac{1}{\sqrt{n}(1 + u^2/n)^n} = \sum a_n b_n(u)
\]

where \( b_n(u) = \frac{(-1)^n}{n^{1/4}(1+u^2/n)^n} \) and \( a_n = 1/n^{1/4} \). Then \( \sum b_n(u) \) converges by the alternating series theorem, and \( a_n \searrow 0 \). Hence it converges. Notice we really couldn’t apply Weierstrass M-test. Now recall Theorem 21 in Buck: Let \( \sum u_n(x) \) converge to \( F(x) \) for \( x \in [a,b] \). let \( u_n'(x) \) exist and be continuous in \( x \in [a,b] \) and let \( \sum u_n'(x) \) converge uniformly on \( [a,b] \). Then

\[
\sum u_n'(x) = F'(x)
\]

So we need to differentiate each term and see if it converges uniformly. Thus

\[
\frac{d}{du} \left( \frac{(-1)^n \left( 1 + \frac{u^2}{n} \right)^{-n}}{\sqrt{n}} \right) = \frac{(-1)^{n+1}2u}{\sqrt{n}(1 + u^2/n)^{n+1}}
\]

Again we will use Abel’s theorem since

\[
\sum \frac{(-1)^{n+1}2u}{(1 + u^2/n)^{n+1}n^{1/4}} < \infty
\]

by the alternating series theorem, and \( 1/n^{1/4} \searrow 0 \). Thus it converges uniformly and so \( \phi \) is differentiable on \( \mathbb{R} \).
Problem 5

So notice we can parameterize $c_1(t)$ in polar coordinates by

$$r_1(t) = t = \frac{\theta}{\pi}$$

and

$$r_2(t) = t + 1 = \frac{\theta}{\pi} + 1$$

for $\theta : 0 \to 4\pi$. Let $L : (x_1, y_1) - (x_2, y_2)$ be the line segment from $(x_1, y_1)$ to $(x_2, y_2)$. So graphically, $\Omega$ is the spiral curve with thickness 1 that starts at $L : (-2, 0) - (-1, 0)$ and spirals downward counterclockwise to the line $L : (0, -3/2) - (0, -5/2)$. Then as we continue in the counterclockwise direction, we go to the line $L : (2, 0) - (3, 0)$, then up to $L : (0, 5/2) - (0, 7/2)$, then over to $L : (-4, 0) - (-3, 0)$, down to $L : (0, -7/2) - (0, -9/2)$ then finally up to $L : (4, 0) - (5, 0)$.

The area of $\Omega$ we can compute easily since the equation for polar integration is

$$\int_{\theta_1}^{\theta_2} \frac{1}{2} (r_1(\theta))^2 - (r_2(\theta))^2 \, d\theta$$

so we have

$$A(\Omega) = \int_{0}^{4\pi} \left( \frac{\theta}{\pi} + 1 \right)^2 - \left( \frac{\theta}{\pi} \right)^2 \, d\theta = \int_{0}^{4\pi} \frac{1}{2} \left( \frac{2\theta}{\pi} + 1 \right) \, d\theta = 10\pi$$

Sept 1997 Complex Analysis

Problem 1

First notice for $\alpha \in \mathbb{C}$,

$$\left| \frac{z - \alpha}{1 - \overline{\alpha}z} \right| = 1$$

This can be seen by letting $z = x + iy$ and $\alpha = a + ib$ and showing $|z - \alpha| = |1 - \overline{\alpha}z|$. So let $g(z) = f(Rz)$. Then $g$ is analytic in $|z| < 1$ and continuous for $|z| \leq 1$. So $g$ has zeros at $z_1/R, z_2/R, ..., z_N/R$ with

$$0 < \left| \frac{z_i}{R} \right| < 1$$

Then let

$$h(z) = \frac{g(z)}{\prod_{i=1}^{N} \left( \frac{z - z_i}{1 - \overline{z_i}/R} \right)}$$

has no zeros inside the unit circle and $|h(z)| = 1$ on $|z| = 1$. Thus $|h(z)| = 1$ on $|z| \leq 1$ implies that $h(z)$ is constant on $|z| \leq 1$. So $h(z) = e^{i\beta}$. So
\[ g(z) = \prod_{i=1}^{N} \left( \frac{z - \frac{1}{N}}{1 - \frac{1}{N} z} \right) e^{i\beta} \]

Therefore

\[ f(z) = \prod_{i=1}^{N} \left( \frac{z - z_i}{1 - \frac{1}{z_i} z} \right) e^{i\beta} \]

**Problem 2**

We first map \( D = \{ z : |z| < 1, 0 < \arg(z) < 2\pi/3 \} \) to the upper-half unit semi-circle by \( w_1 = z^{3/2} \). Then we map that to the first quadrant by

\[ w_2 = i \left( \frac{1 - z^{3/2}}{1 + z^{3/2}} \right) \]

then map the first quadrant to the UHP by

\[ w_3 = -\left( \frac{1 - z^{3/2}}{1 + z^{3/2}} \right)^2 \]

Finally we map the UHP to the interior of the unit circle by

\[ w_f = \frac{-\left( \frac{1 - z^{3/2}}{1 + z^{3/2}} \right)^2 - i}{\left( \frac{1 - z^{3/2}}{1 + z^{3/2}} \right)^2 + i} \]

**Problem 3.a**

Let \( \gamma \) be the rectangular contour from \(-N\) to \(N\) to \(N + i2\pi\) to \(-N + i2\pi\) to \(-N\). Then let

\[ f(z) = \frac{e^{az}}{1 + e^{z}} \]

Then by the residue theorem, we have

\[ \int_{\gamma} f(z) dz = 2\pi i \text{Res}(f, i) = 2\pi i \frac{e^{ai\pi}}{e^{i\pi}} = \frac{2\pi i}{e^{i\pi(1-\epsilon)}} \]

now on the contour where \( z = N + iy \) for \( y : 0 \to 2\pi \), we have

\[ \left| i \int_{0}^{2\pi} \frac{e^{aN+ayi}}{1 + e^{N+iy}} dy \right| \leq 2\pi e^{aN} \frac{e^{aN}}{e^{N} - 1} \to 0 \]

since \( a < 1 \). Likewise on the contour where \( z = -N + iy \) for \( y : 2\pi \to 0 \), we have

\[ \left| i \int_{2\pi}^{0} \frac{e^{-aN+ayi}}{1 + e^{-N+iy}} dy \right| \leq 2\pi e^{-aN} \frac{e^{-aN}}{e^{-N} - 1} \to 0 \]
since \(a > 0\). Thus as \(N \to \infty\), we have

\[
\int_{-\infty}^{\infty} \frac{e^{az}}{1 + e^z} \, dz + \int_{-\infty}^{\infty} \frac{e^{a(x+2\pi i)}}{1 + e^{x+2\pi i}} \, dx = (1 - e^{a2\pi i}) \int_{-\infty}^{\infty} \frac{e^{az}}{1 + e^z} \, dz = \frac{2\pi i}{e^{i\pi(1-a)}}
\]

therefore

\[
\int_{-\infty}^{\infty} \frac{e^{az}}{1 + e^z} \, dz = \frac{2\pi i}{e^{i\pi(1-a)}}
\]

**Problem 3.b**

The problem must be a typo, because it does not converge. Now let's assume the problem is to solve

\[
\int_{0}^{\infty} \frac{x^b \log x}{1 + x^2} \, dx
\]

Then the \(\gamma\) be the key hold contour, and

\[
f(z) = \frac{z^b \log z}{1 + z^2}
\]

where we define \(\log z = \ln |z| + i \arg z\) where \(0 < \arg z < 2\pi\). Hence we are going to cut out the nonnegative real axis. Then by the Residue Theorem, we have

\[
\int_{\gamma} f(z) \, dz = 2\pi i \text{Res}(f, i) + 2\pi i \text{Res}(f, -i) = \pi e^{i\pi b/2} i\pi/2 - \pi e^{b3\pi/2} i3\pi/2
\]

Now notice on \(C_R\), we have

\[
\left|\int_{C_R} f(z) \, dz\right| \leq 2\pi R \frac{R^b \log R}{R^2 - 1} \to 0
\]
as \(R \to \infty\) and

\[
\left|\int_{C_{\epsilon}} f(z) \, dz\right| \leq 2\pi \epsilon \frac{\epsilon^b \log \epsilon}{\epsilon^2 - 1} \to 0
\]

Thus for \(z = re^{i\delta}\) as \(\delta \to 0\), we have

\[
\lim_{\delta \to 0} \int_{0}^{\infty} \frac{r^b e^{b\delta} (\log r + i\delta)^e r^i \delta}{1 + r^2 e^{i2\delta}} \, dr = \int_{0}^{\infty} \frac{r^b \log r}{1 + r^2} \, dr
\]

and on \(z = re^{i(2\pi - \delta)}\), we have

\[
\lim_{\delta \to 0} \int_{0}^{\infty} \frac{r^b e^{i(2\pi - b\delta) (\log r + i(2\pi - \delta))} e^{i\delta}}{1 + r^2 e^{i2\delta} + 2\delta} \, dr = \int_{0}^{\infty} \frac{r^b \log r}{1 + r^2} \, dr - \int_{0}^{\infty} \frac{r^b \log r}{1 + r^2} \, dr - \int_{0}^{\infty} \frac{r^b e^{i2\pi} e^{2\pi}}{1 + r^2} \, dr
\]

By a similar argument we can see that
Thus
\[
\int_0^\infty \frac{r^b e^{i b 2\pi i}}{1 + r^2} dr = \frac{\pi e^{i b \pi/2} - \pi e^{i b 3\pi/2}}{1 - e^{i 2\pi b}}
\]

\[
\int_0^\infty \frac{x^b \log x}{1 + x^2} dx = \frac{\pi e^{i b \pi/2} - \pi e^{i b 3\pi/2} + \left( \frac{i \pi^2 e^{i \pi b/2}}{2} - \frac{i \pi^2 e^{i 3\pi b/2}}{2} \right) (1 - e^{i 2\pi b})}{\pi e^{i b \pi/2} - \pi e^{i b 3\pi/2}}
\]

Sept 1997 Linear Algebra

Problem 1.a

Using Gaussian elimination, we have

\[
\begin{bmatrix}
3 & 3 & 2 & 0 \\
3 & 6 & 3 & 1 \\
3 & 0 & 1 & 1 \\
\end{bmatrix}
\]

\[
\begin{array}{cccc}
3 & 3 & 2 & 0 \\
0 & 3 & 1 & 1 \\
0 & -3 & -1 & 1 \\
\end{array}
\]

\[
\begin{bmatrix}
3 & 3 & 2 & 0 \\
0 & 3 & 1 & 1 \\
0 & 0 & 0 & 2 \\
\end{bmatrix}
\]

Hence we have 0 = 2, which implies there are no solutions to the system.

Problem 1.b

Using Gaussian elimination, we have

\[
\begin{bmatrix}
3 & 3 & 2 & 0 \\
3 & 6 & 3 & 0 \\
3 & 0 & 1 & 0 \\
\end{bmatrix}
\]

\[
\begin{array}{cccc}
3 & 3 & 2 & 0 \\
0 & 3 & 1 & 0 \\
0 & -3 & -1 & 0 \\
\end{array}
\]

\[
\begin{bmatrix}
3 & 3 & 2 & 0 \\
0 & 3 & 1 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Hence we have \( z = t \) and \( x = y = -t/3 \) for \( t \in \mathbb{R} \).

Problem 2.a

we have \( V = S((1, 0, 1)^T, (1, 1, 0)^T) \). Then let \( u_1 = (1/\sqrt{2})(1, 0, 1)^T \) and
\[ u_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - ((1,1,0)^T, (1/\sqrt{2})(1,0,1)^T) \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{3/2}} \begin{pmatrix} 1/2 \\ 1 \\ -1/2 \end{pmatrix} \]

Problem 2.b

\[ V^\perp = S((1,-1,-1)^T) \]

Problem 2.c

\[ V = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \]

So \( P = V(V^T V)^{-1} V^T \)

Now

\[ P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix} \]

So \( P(3,0,0)^T = (2,1,1)^T \). Hence \( \sqrt{9-6} = \sqrt{3} \) is the distance from \( V \).

Problem 2.d

The perpendicular matrix of \( v \) is \( u \) where

\[ u = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \]

Hence

\[ A = I - \frac{2}{3} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{pmatrix} \]

Problem 3.a

Notice

\[ \det(A - \lambda I) = \det \begin{pmatrix} s + 1 - \lambda & 1 - t \\ -1 - t & s - 1 - \lambda \end{pmatrix} = \lambda^2 - \lambda 2s + s^2 - 1 + 1 - t^2 \]

using the quadratic equation, we have the two roots

\[ \lambda_1 = s + t \quad \text{and} \quad \lambda_2 = s - t \]
Problem 3.b

Remember that $A$ is diagonalizable if it has two linearly independent eigenvectors. Since different eigenvalues correspond to linearly independent eigenvectors, we need $s + t = s - t$ (since we don’t want to diagonalize $A$. Hence $t = 0$. Then we have

$$A = \begin{pmatrix} s + 1 & 1 \\ -1 & s - 1 \end{pmatrix}$$

Now the corresponding eigenvector(s) is

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Thus there is only 1 linearly independent eigenvector $v = (1, -1)$ that satisfies above. So $s \in R$ and $t = 0$ suffices.

Problem 3.c

If $t \neq 0$ then $A$ is diagonalizable. Hence

$$A = M \begin{pmatrix} s + t & 0 \\ 0 & s - t \end{pmatrix} M^{-1}$$

where $M = (v_1, v_2)$, and $v_1, v_2$ are the two linearly independent eigenvectors for $s + t$ and $s - t$ respectively. Now

$$A^k = M \begin{pmatrix} s + t & 0 \\ 0 & s - t \end{pmatrix}^k M^{-1} = M \begin{pmatrix} (s + t)^k & 0 \\ 0 & (s - t)^k \end{pmatrix} M^{-1}$$

Hence $\lim_k A^k$ exists when $|s + t| < 1$ and $|s - t| < 1$. Now if $t = 0$, then

$$A = \begin{pmatrix} s + 1 & 1 \\ -1 & s - 1 \end{pmatrix}$$

and $A$ is not diagonalizable. So the Jordan Form is

$$A = \begin{pmatrix} 1 & x \\ -1 & y \end{pmatrix} \begin{pmatrix} s & 1 \\ 0 & s \end{pmatrix} \begin{pmatrix} 1 & x \\ -1 & y \end{pmatrix}^{-1}$$

To solve for $v_2 = (x, y)$, we have

$$A v_2 = v_1 + s v_2 \Rightarrow (A - sI)v_2 = v_1$$

which implies

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Hence we have $v_2 = (0, 1)^T$. So
\[ A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} s & 1 \\ 0 & s \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^{-1} \]

which implies

\[ A^k = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} s^k & ks^{k-1} \\ 0 & s^k \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^{-1} \]

Hence \(|s - t| < \) and \(|s + t| < \) suffices. Note that for any square matrix \(A\), if \(|\lambda_i| < 1\) for all \(i\), then \(A^k\) converges.

**Problem 5.a**

We know that \(\text{Rank}(A) \leq n < m\) and \(\text{Rank}(B) \leq n < m\). Also

\( \text{Rank}(AB) \leq \min(\text{Rank}(A), \text{Rank}(B)) \leq n < m \)

Also since

\( \text{Rank}(AB) + n(AB) = m \)

this implies \(n(AB) \neq 0\). Hence \(AB\) is singular. Note for rectangular \(A\), \(A^TA\) is nonsingular if the columns of \(A\) are linearly independent. Here we are in the opposite case.

**Problem 5.b**

Assume

\( \text{Rank}(A + B) > \text{Rank}(A) + \text{Rank}(B) \)

Now let \((a_1, \ldots, a_n)\) be the columns of \(A\) and \((b_1, \ldots, b_n)\) be the columns of \(B\). Let \(\text{Rank}(A) = i\) and \(\text{Rank}(B) = j\). WLOG let \(a_1, \ldots, a_i\) be the linearly independent column vectors of \(A\) and \(b_1, \ldots, b_j\) be the linearly independent column vectors of \(B\). Now if \(\text{Rank}(A + B) > \text{Rank}(A) + \text{Rank}(B)\), then \(\exists\) column \(x \in A + B\), such that \(s \not\in S(a_1, \ldots, a_i, b_1, \ldots, b_j)\). But then we have

\( x \in S(a_{i+1}, \ldots, a_n, b_{j+1}, \ldots, b_n) \subset S(a_1, \ldots, a_i, b_1, \ldots, b_j) \)

and so we have a contradiction.

**Problem 5.c**

We have \(A\) and \(B\) \(m \times n\) matrices, and

\( \text{Rank}(A + B) + n(A + B) = n \)

By above we know that \(\text{Rank}(A + B) \leq \text{Rank}(A) + \text{Rank}(B)\). Thus

\[ n(A+B) = n - \text{Rank}(A+B) \geq n - \text{Rank}(A) - \text{Rank}(B) = n - (n - n(A)) - (n - n(B)) = n(A) + n(B) - n \]

-35
Jan 1998 Advanced Calculus

Problem 1

So we have \( g(\vec{x}) = C = x_1x_2 \cdots x_n \) and

\[
f(\vec{x}) = \frac{a_1}{x_1} + \frac{a_2}{x_2} + \cdots + \frac{a_n}{x_n}
\]

and so by the Lagrange Multiplier theorem, we have \( \nabla f = \lambda \nabla g \) which implies

\[
\left( -\frac{a_1}{x_1^2}, -\frac{a_2}{x_2^2}, \ldots, -\frac{a_n}{x_n^2} \right) = \lambda \left( x_2x_3 \cdots x_n, x_1x_3x_4 \cdots x_n, \ldots, x_1x_2 \cdots x_{n-1} \right)
\]

Which implies

\[
-\frac{a_1}{x_1^2} = -\frac{a_2}{x_2^2} = \cdots = -\frac{a_n}{x_n^2}
\]

which implies

\[
\frac{a_1}{x_1} = \frac{a_2}{x_2} = \cdots = \frac{a_n}{x_n}
\]

So let \( x_1 = t \). Then \( x_2 = (a_2/a_1)t, x_3 = (a_3/a_1)t, \ldots, x_n = (a_n/a_1)t \). Since \( x_1x_2 \cdots x_n = C \), we have

\[
\frac{t^n(a_2a_3 \cdots a_n)}{a_1^{n-1}} = C
\]

which implies

\[
t = \frac{C^{1/n}a_1}{(a_1a_2 \cdots a_n)^{1/n}} = x_1
\]

\[
x_2 = \frac{C^{1/n}a_2}{(a_1a_2 \cdots a_n)^{1/n}}
\]

\[
\vdots
\]

\[
x_n = \frac{C^{1/n}a_n}{(a_1a_2 \cdots a_n)^{1/n}}
\]

So then the extrema is

\[
f(\vec{x}) = \frac{n(a_1a_2 \cdots a_n)^{1/n}}{C^{1/n}}
\]

Now this is clearly the minimum, since for \( \vec{b} = (b_1, b_2, \ldots, b_n) \) where \( b_i = C^{1/n} \forall i \), we have \( \vec{b} \in S \) and

\[
f(\vec{b}) = \frac{a_1 + a_2 + \cdots + a_n}{C^{1/n}}
\]
and since \( a_i > 0 \) for all \( i \), we know that the geometric mean of \( a_i \) is less than or equal to the arithmetic mean. Hence

\[
(a_1 a_2 \cdots a_n)^{1/n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n}
\]

So

\[
f(\overline{x}) = \frac{n(a_1 a_2 \cdots a_n)^{1/n}}{C^{1/n}} \leq \frac{n}{C^{1/n}} \frac{a_1 + a_2 + \cdots + a_n}{n} = \frac{a_1 + a_2 + \cdots + a_n}{C^{1/n}} = f(\overline{b})
\]

So the extrema is indeed the minimum.

**Problem 2**

\[
\left| \sum_{j=0}^{n-1} \frac{f(j/n)}{n} - \int_0^1 f(x)dx \right|
\]

\[
= \left| \left( \frac{f(0)}{n} - \int_0^{1/n} f(x)dx \right) + \left( \frac{f(1/n)}{n} - \int_{1/n}^{2/n} f(x)dx \right) + \cdots + \left( \frac{f((n-1)/n)}{n} - \int_{(n-1)/n}^1 f(x)dx \right) \right|
\]

by triangle inequality, we have

\[
\leq \left| \frac{f(0)}{n} - \int_0^{1/n} f(x)dx \right| + \left| \frac{f(1/n)}{n} - \int_{1/n}^{2/n} f(x)dx \right| + \cdots + \left| \frac{f((n-1)/n)}{n} - \int_{(n-1)/n}^1 f(x)dx \right|
\]

Now let \( i \) be the maximum of partials. Then notice

\[
\left| \frac{f(i/n)}{n} - \int_{i/n}^{(i+1)/n} f(x)dx \right| \leq \left( \frac{M}{1/n} \right) \frac{1}{n^2}
\]

Thus

\[
\left| \sum_{j=0}^{n-1} \frac{f(j/n)}{n} - \int_0^1 f(x)dx \right| \leq n \left( \frac{M}{1/n} \right) \frac{1}{n^2} = \frac{M}{2n}
\]

**Problem 3**

Notice that the center of the circle is at \((t/3, t/3, t/3)\). Now the distance to the origin is \(t/\sqrt{3}\). Since the sphere has radius 1, the distance from the center of \(S\) to the edge is

\[
\sqrt{1 - \frac{t^2}{3}}
\]

Now to take advantage of \(x^2 + y^2 + z^2 = \rho^2\), let’s define \(\rho\) in terms of \(r\) where \(r\) is the radial distance of \(S\). Again by geometry, we have on \(S\)
\[ x^2 + y^2 + z^2 = r^2 + \frac{t^2}{3} \]

Hence
\[
\int \int_S (1 - (x^2 + y^2 + z^2))dA = \int_0^{2\pi} \int_0^{\sqrt{\frac{1-t^2}{3}}} (1 - r^2 - \frac{t^2}{3})rdrd\theta = \frac{\pi}{18}(3 - t^2)^2
\]

**Problem 4.a**

Notice that
\[ (1 - \cos 1/x) \sim \frac{1}{x^2} \]
indeed
\[
\lim_{x \to \infty} \frac{1 - \cos(1/x)}{1/x^2} = \lim_{u \to 0} \frac{1 - \cos u}{u^2} = \lim_{u \to 0} \frac{\sin u}{2u} = \lim_{u \to 0} \frac{\cos u}{2} = \frac{1}{2} \leq 1
\]
So for large enough \( N \) we have for all \( n > N \)
\[ 1 - \cos(1/x) \leq \frac{1}{x^2} \]
Hence
\[
\int_1^{\infty} x(1 - \cos(1/x))^\beta \leq \int_1^N x(1 - \cos(1/x))^\beta dx + \int_N^{\infty} x \left(\frac{1}{x^2}\right)^\beta dx = K \int_N^{\infty} \frac{1}{x^{2\beta-1}} dx
\]
Which converges for \( \beta > 1 \). Now notice
\[
\lim_{x \to \infty} \frac{1 - \cos(1/x)}{(1/4)(1/x^2)} = \lim_{u \to 0} \frac{1 - \cos u}{(1/4)u^2} = \lim_{u \to 0} \frac{\sin u}{(1/2)u} = \lim_{u \to 0} \frac{\cos u}{1/2} = 2 \geq 1
\]
So for large enough \( x \) we have
\[ 1 - \cos(1/x) \geq \frac{1}{4} \left(\frac{1}{x^2}\right) \]
Hence
\[
\int_1^{\infty} x(1 - \cos(1/x))^\beta dx \geq \int_1^N x(1 - \cos(1/x))^\beta dx + \frac{1}{4} \int_N^{\infty} \frac{1}{x^{2\beta-1}} dx
\]
which diverges for \( \beta \leq 1 \).
Problem 4.b

Notice
\[ \left| \frac{x^j}{j^2(1 + x^j)} \right| \leq \frac{1}{j^2} = M_j \]

and \( \sum M_j < \infty \). SO by the Weierstrass M-test, we have
\[ \sum_{j=1}^{\infty} \frac{x^j}{j^2(1 + x^j)} \]
converges uniformly for \( x \geq 0 \). Now for \( |x| \leq a < 1 \), we have
\[ \left| \frac{x^j}{j^2(1 + x^j)} \right| \leq \frac{|x^j|}{j^2(|x|^j - 1)} \leq \frac{a}{j^2(1 - a)} = M_j \]
\[ \sum M_j < \infty \] and by the Weierstrass M-test, it converges uniformly.

Problem 5.a

since \( 0 \leq 2|u| \), we have
\[ 1 + u^2 \leq 1 + 2|u| + u^2 \Rightarrow 1 + u^2 \leq (1 + |u|)^2 \Rightarrow \sqrt{1 + u^2} \leq 1 + |u| \]

Jan 1998 Complex Variables

Problem 2

Let \( \gamma \) be the contour with outer radius \( R \) and inner radius \( \epsilon \) in the upper half plane. Also let
\( f(z) = \frac{1}{(1+z^2)^{1/2}} \). Then by the residue theorem, we have
\[ \int_\gamma f(z) \, dz = 2\pi i \frac{1}{2i\sqrt{i}} = \pi \sqrt{i} = \frac{\pi}{\sqrt{2}} (1 - i) \]

Notice on the outer radius
\[ \left| \int_{|z|=R} f(z) \, dz \right| \leq \pi R \frac{1}{\sqrt{R(R^2 - 1)}} \to 0 \]
as \( R \to \infty \). Likewise on the inner radius we have
\[ \left| \int_{|z|=\epsilon} f(z) \, dz \right| \leq \pi \sqrt{\frac{\epsilon}{1 - \epsilon^2}} \to 0 \]
as \( \epsilon \to 0 \). Hence as \( R \to \infty \) and \( \epsilon \to 0 \), we have
\[ \int_{-\infty}^{0} f(z) \, dz + \int_{0}^{\infty} f(z) \, dz = (1 - i) \int_{0}^{\infty} \frac{dz}{(1 + z^2)^{1/2}} = \frac{\pi}{\sqrt{2}} (1 - i) \]
Hence
\[ \int_0^\infty \frac{dx}{(1 + x^2)\sqrt{x}} = \frac{\pi}{\sqrt{2}} \]

**Problem 3.a**

Let
\[ g(z) = \frac{(z - z_1)(z - z_2) \cdots (z - z_m) f(z)}{z} \]

then \( g \) is an entire function. So it has a taylor series about \( z = 0 \)
\[ g(z) = a_0 + a_1 z + a_2 z^2 + \cdots \]

Also we know that
\[ \lim_{z \to 0} |g(1/z)| < \infty \]

Hence \( g(z) = a_0 \). Therefore
\[ f(z) = \frac{a_0 z}{(z - z_1) \cdots (z - z_m)} \]

**Problem 3.b**

By part (a), \( \deg(P) = 1 \) and \( \deg(Q) = m \).

**Problem 4**

We start by defining
\[ (g(z))^{2/3} = e^{\frac{2}{3} \log(g(z))} = e^{\frac{2}{3} (\ln(|g(z)|) + i \arg(g(z)))} \]

where
\[ -\frac{3\pi}{2} < \arg(g(z)) < \frac{\pi}{2} \]

Hence now \( \Phi \) is analytic in the UHP. So notice by Schwartz-Christoffel this is an equilateral triangle. We have
\[ \Phi(-\infty) = 0 \]

and
\[ \Phi(0) = \int_{-\infty}^{0} \frac{dx}{(-x)^{2/3}(1 - x)^{2/3}} = \int_{0}^{\infty} x^{-2/3}(1 + x)^{-2/3} dx > 0 \]

So let
\[
C = \int_{0}^{\infty} x^{-2/3}(1 + x)^{-2/3} \, dx
\]

So \((-\infty, 0] \mapsto [0, C]\). Then for \(x \in (0, 1)\), we have

\[
\Phi'(x) = (-x)^{-2/3}(1 - x)^{-2/3}
\]

and

\[
\arg(\Phi'(x)) = (-\pi)(-2/3) + 0 = 2\pi/3
\]

And for \(x > 1\) we have

\[
\arg(\Phi(x)) = (-\pi)(-2/3) + (-\pi)(-2/3) = 4\pi/3
\]

So we have an equilateral triangle with vertices \((0, C, C/2 + iC\sqrt{3}/2)\). By the orientation of the mapping, the UHP maps to the interior of the triangle.

**Problem 5**

We will use Rouche’s Theorem. Notice for \(|z| = 2n\), we have

\[
\left| P_n(z) - \frac{z^n}{n!} \right| = |P_{n-1}| \leq \sum_{k=0}^{n-1} \frac{(2n)^k}{k!} < \frac{(2n)^n}{n!} = \frac{z^n}{n!}
\]

Hence there are \(n\) zeros inside \(|z| < 2n\). By the fundamental theorem of algebra, this means that all the zeros are inside the circle. Now to see the strict inequality notice

\[
1 + 2 + 2^2 + \cdots + 2^{n-1} < 2^n
\]

Hence

\[
\frac{n^n}{n!} \left(1 + 2 + 2^2 + \cdots + 2^{n-1}\right) < \frac{2^n n^n}{n!}
\]

which implies

\[
\frac{n^n}{n!} + \frac{2n^n}{n!} + \cdots + \frac{2^{n-1}n^n}{n!} < \frac{2^n n^n}{n!}
\]

and

\[
\frac{n^n}{n!} \geq 1 \quad \frac{2n^n}{n!} \geq 2n \quad \cdots \quad \frac{2^{n-1}n^n}{n!} \geq \frac{2^{n-1}n^{n-1}}{(n-1)!}
\]

So

\[
\sum_{k=0}^{n-1} \frac{(2n)^k}{k!} < \frac{(2n)^n}{n!}
\]
Jan 1998 Linear Algebra

Problem 1

Notice we have

\[ \det(A - \lambda I) = \det \begin{pmatrix} 3 - \lambda & 2 \\ 2 & 3 - \lambda \end{pmatrix} = (3 - \lambda)^2 - 4 = (\lambda - 5)(\lambda - 1) \]

Hence \( \lambda_1 = 5 \) and \( \lambda_2 = 1 \). Notice

\[ \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5x \\ 5y \end{pmatrix} \]

Hence we have \( 3x + 2y = 5x \), which implies \( x = y \). Thus our eigenvector \( u_1 = (1, 1) \). Also notice

\[ \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \]

implies \( 3x + 2y = x \), and thus \( x = -y \). Therefore our second eigenvector \( u_2 = (1, -1) \). Likewise for matrix \( B \) we have

\[ \det \begin{pmatrix} 4 - \lambda & 1 \\ 1 & 4 - \lambda \end{pmatrix} = (4 - \lambda)^2 - 1 = (\lambda - 3)(\lambda - 5) \]

Hence for our matrix \( B \), we have eigenvalues \( \lambda_1 = 3 \) and \( \lambda_2 = 5 \). Notice

\[ \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \end{pmatrix} \]

Hence we have \( 4x + y = 3x \), which implies \( x = -y \). Thus our eigenvector \( u_1 = (1, -1) \). Also notice

\[ \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5x \\ 5y \end{pmatrix} \]

implies \( 4x + y = 5x \), and thus \( x = y \). Therefore our second eigenvector \( u_2 = (1, 1) \). Now notice

\[ x = \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \frac{5}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \]

Hence

\[ A^N x = A^N \left( \frac{5}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) = \frac{5^{N+1}}{2} \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) = \left( \frac{5^{N+1} + 1}{5^{N+1} - 1} \right) \]

and

\[ B^N x = B^N \left( \frac{5}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) = \frac{5^{N+1}}{2} \left( \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{3}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right) = \left( \frac{5^{N+1} + 3^N}{5^{N+1} - 3^N} \right) \]

Therefore

\[-42\]
\[ ||A^N x|| = \frac{\sqrt{2 \cdot 5^{2N+2} + 2}}{2} \]

and

\[ ||B^N x|| = \frac{\sqrt{2 \cdot 5^{2N+2} + 2 \cdot 3^{2N}}}{2} \]

Finally

\[
\lim_{n \to \infty} \frac{||A^N x||}{||B^N x||} = 1
\]

**Problem 1.b**

By part (a), we can see that \( ||B^N x|| \) is greater than \( ||A^N x|| \) for large \( N \).

**Problem 2.a**

Notice that \( \text{det}(A - \lambda I) = (4 - \lambda)(5 - \lambda)^2 \). Thus we know that there are 2 linearly independent eigenvectors corresponding to 4 and 5. Notice

\[
(A - 5I) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & 0 & -2 \\ 2 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

Hence we have \(-x - 2z = 0\) and \(2x + 4z = 0\). So notice we have two linearly independent eigenvectors \( v_1 \) and \( v_2 \) with eigenvalues of 5

\[
v_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}
\]

Thus \( A \) is diagonalizable and

\[
A = \begin{pmatrix}
-\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\
\frac{2}{\sqrt{5}} & 0 & 1 \\
0 & -\frac{1}{\sqrt{5}} & 0
\end{pmatrix}
\begin{pmatrix}
4 & 0 & 0 \\
0 & 5 & 0 \\
0 & 0 & 5
\end{pmatrix}
\begin{pmatrix}
-\sqrt{5} & 0 & -2\sqrt{5} \\
0 & 0 & -\sqrt{5} \\
2 & 1 & 5
\end{pmatrix} = SDS^{-1}
\]

**Problem 2.b**

Notice we have

\[
\text{det}(B - \lambda I) = (5 - \lambda)^2(4 - \lambda)
\]

By inspection we can see that \( B \) is not diagonalizable, and hence the Jordan matrix is

\[
J = \begin{pmatrix}
5 & 1 & 0 \\
0 & 5 & 0 \\
0 & 0 & 4
\end{pmatrix}
\]
Hence for $M = (v_1, v_2, v_3)$, $v_1$ and $v_3$ are eigenvectors. Now for $\lambda = 5$, we have

$$B - 5I = \begin{pmatrix} -1 & 0 & -6 \\ 6 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

which implies $-x - 6z = 0$ and $6x + 4z = 0$, which implies $v_1 = (0, 1, 0)^T$. Now for $\lambda = 4$ we have

$$\begin{pmatrix} 0 & 0 & -6 \\ 6 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

which implies $-6z = 0$, $6x + y + 4z = 0$, and $z = 0$. Thus $v = 1/\sqrt{37}(-1, 6, 0)^T$. Now notice $Av_2 = v_1 + 5v_2$. Hence $(A - 5I)v_2 = v_1$,

$$\begin{pmatrix} -1 & 0 & -6 \\ 6 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Hence $x = 3/16$, $y = -1/32$, and $z = -1/32$. Hence

$$A = MJM^{-1} = \begin{pmatrix} 0 & 3 & -1 \\ 1/16 & -1/\sqrt{37} & 5/37 \\ 0 & 3/32 & 0 \end{pmatrix} \begin{pmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 6 & 1 & 35 \\ 0 & 0 & -32 \\ -\sqrt{37} & 0 & -6\sqrt{37} \end{pmatrix}$$

Problem 3.a

$$\begin{pmatrix} a & b & c & d \\ 1/2 & e & f & g \\ 1/2 & 1/2 & h & i \\ 1/2 & 1/2 & 1/2 & j \end{pmatrix} \begin{pmatrix} a & 1/2 & 1/2 & 1/2 \\ b & e & 1/2 & 1/2 \\ c & f & h & 1/2 \\ d & g & i & j \end{pmatrix}$$

Clearly we must have $j = 1/2$. Then

$$1/2h^2 + i^2 = 1 \quad \text{and} \quad 1/2 + (1/2)h + (1/2)i = 0$$

Hence we must have $h = -1/2$ and $i = -1/2$. Then

$$1/4 + (1/2)e + (1/2)f + (1/2)g = 0 \quad \text{and} \quad 1/4 + (1/2)e - (1/2)f - (1/2)g = 0 \quad \text{and} \quad 1/4 + e^2 + f^2 + g^2 = 1$$

which implies we have $e = -1/2$, $f = 1/2$ and $g = -1/2$. Finally

$$(1/2)a + (1/2)b + (1/2)c + (1/2)d = 0 \quad \text{and} \quad (1/2)a + (1/2)b - (1/2)c - (1/2)d = 0$$

$$(1/2)a - (1/2)b + (1/2)c - (1/2)d = 0 \quad \text{and} \quad a^2 + b^2 + c^2 + d^2 = 1$$

which implies that $a = d = -1/2$ and $b = c = 1/2$. 

-44
Problem 3.b
By part (a) we can see that it is unique.

Problem 5.a
Clearly if $|\lambda_i| < 1$ with eigenvector $v_i$, then
\[
\frac{||A^N x||}{||x||} = \frac{||(\lambda_i)^N x||}{||x||} = \frac{|\lambda_i|^N ||x||}{||x||} = |\lambda_i|^N < c_1
\]
for large enough $N$, and thus we have a contradiction. Now if $|\lambda_j| > 1$ with eigenvector $v_j$ such that
\[
\frac{||A^N v_j||}{||v_j||} = \frac{||(\lambda_j)^N v_j||}{||v_j||} = |\lambda_j|^N > c_2
\]
for large enough $N$, which implies we have a contradiction. Hence $|\lambda_i| = 1$ for all $i$.

Problem 5.b
Assume $A$ has the Jordan form $A = MJM^{-1}$, and assume that $J$ has a Jordan block (i.e. $J$ is not diagonal). The $\exists x_1, x_2$ such that $x_1 \neq x_2$ and
\[
Ax_1 = \lambda_1 x_1 \quad \text{and} \quad Ax_2 = x_1 + \lambda_1 x_2
\]
Thus
\[
A^n x_2 = N\lambda_1^{N-1} x_1 + \lambda_1^N x_2
\]
Thus
\[
\frac{||A^N x_2||}{||x_2||} = \frac{||N\lambda_1^{N-1} x_1 + x_2||}{||x_2||} \to \infty
\]
as $N \to \infty$. Hence we have a contradiction which implies that $J$ has no Jordan blocks. Thus $A = MJM^{-1}$ and $J$ is unitary by part (a).

Sept 1998 Advanced Calculus

Problem 1.a
Notice that $\frac{n^2 + 1}{n^3 \log n} \to 0$ for $n \geq 2$. Hence by the alternating series theorem,
\[
\sum_{n=1}^{\infty} (-1)^n \frac{n^2 + 1}{n^3 \log n}
\]
converges. However it does not converge absolutely. Notice
\[
\sum_{n=1}^{\infty} \frac{n^2 + 1}{n^3 \log n} \leq 2 \sum_{n=1}^{\infty} \frac{n^2}{n^3 \log n} = \sum_{n=1}^{\infty} \frac{1}{n \log n} \to \infty
\]
Problem 1.b

be letting \( u = 1/x \) and \( du = -x^{-2} \), we have

\[
\int_{0}^{\infty} \frac{x^2 e^{1/x}}{1 + x^4} \, dx = \int_{u=\infty}^{0} \frac{-u^{-4}e^u}{1 + u^{-4}} \, du = \int_{u=0}^{\infty} \frac{e^u}{u^4 + 1} \, du
\]

we know that \( \int_{0}^{\infty} \frac{e^u}{u^4 + 1} \, du \) diverges since \( \frac{e^u}{u^4 + 1} \to \infty \) as \( u \to \infty \). Hence it diverges.

Problem 2

We have

\[
\frac{df}{dx} = \frac{dG}{du} \frac{du}{dx}
\]

\[
\frac{d^2 f}{dx^2} = \frac{d^2 G}{du^2} \left( \frac{du}{dx} \right)^2 + \frac{d^2 u}{dx^2} \frac{dG}{du}
\]

and

\[
\frac{df}{dy} = \frac{dG}{du} \frac{du}{dy}
\]

\[
\frac{d^2 f}{dy^2} = \frac{d^2 G}{du^2} \left( \frac{du}{dy} \right)^2 + \frac{d^2 u}{dy^2} \frac{dG}{du}
\]

So

\[
\frac{d^2 f}{dx^2} + \frac{d^2 f}{dy^2} = \frac{d^2 G}{du^2} (4u) + 2 \frac{dG}{du} = H(u)
\]

Problem 3.a

Let

\[
L(m) = \prod_{n=2}^{m} \left[ (1 + a_n)e^{-a_n + a_n^2/2} \right]
\]

then

\[
\log(L(m)) = \sum_{n=2}^{\infty} \left( \log(1 + a_n) - a_n + a_n^2/2 \right)
\]

Now we are given that \( \sum |a_n|^3 < \infty \). Notice that \( a_n \to 0 \). So WLOG, we can assume that \( |a_n| < 1 \) for all \( n \), since otherwise we can just look at the tail. So recall the Taylor expansion

\[
\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \pm \cdots
\]

Let \( E(x) = -x^4/4 \pm \cdots \). Then
\[
\lim_{x \to 0} \frac{E(x)}{x^3} = 0
\]

So for \(x < \delta\), \(E(x) < x^3\). So

\[
\log(L(m)) = \sum_{n=2}^{m} a_n - \frac{a_n^2}{2} + \frac{a_n^3}{3} + E(a_n) - a_n + \frac{a_n^2}{2} = \sum_{n=2}^{m} \frac{a_n^3}{3} + E(a_n)
\]

So now we \(N\) such that \(n > N\) we have \(a_n < \delta\). Then

\[
\leq \left| \sum_{n=2}^{N} \frac{a_n^3}{3} + E_n(a_n) \right| + \left| \sum_{N+1}^{m} \frac{a_n^3}{3} + a_n \right|
\]

which converges since \(\text{sum}|a_n|^3 < \infty\).

**Problem 3.b**

Again let’s take the log.

\[
L(m) = \prod_{n=2}^{m} \left[ 1 + \frac{(-1)^n}{\sqrt{n}} \right]
\]

Hence

\[
\log(L(m)) = \sum_{n=2}^{\infty} \log \left( 1 + \frac{(-1)^n}{\sqrt{n}} \right) = \sum \left( \frac{(-1)^n}{\sqrt{n}} - \frac{1}{2n} + \frac{(-1)^{2n}}{2n\sqrt{n}} + E \left( \frac{(-1)^n}{\sqrt{n}} \right) \right)
\]

Notice that the first, third, and fourth term converges, but the second term diverges. Hence \(\lim_{m} \log(L(m)) = -\infty\), which implies \(\lim_{m} L(m) = 0\).

**Problem 4**

Notice for \(y = \sqrt{nx}\), we have

\[
\lim_{n \to \infty} \sqrt{n} \int_{-1}^{1} (1 - x^2)^n dx = \lim_{n \to \infty} \int_{-\sqrt{n}}^{\sqrt{n}} \left( 1 - \frac{y^2}{n} \right)^n dy = \lim_{n \to \infty} \int_{-\infty}^{\infty} \chi_{[-\sqrt{n},\sqrt{n}]} \left( 1 - \frac{y^2}{n} \right)^n dy
\]

Now I claim that I can apply the dominated converges theorem, since for all \(n\)

\[
\left| \chi_{[-\sqrt{n},\sqrt{n}]} \left( 1 - \frac{y^2}{n} \right)^n \right| \leq e^{-y^2}
\]

This can be shown explicitly by noticing

\[
\log(1 - x) \leq -x
\]

Thus by DCT

\[
= \int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}
\]

-47
Problem 5.a
Not true. Let \( f_n(x) = \sin(nx)/n \). Then pointwise \( f_n(x) \to 0 \). But \( f'_n(x) = \cos(nx) \neq 0 \).

Sept 1998 Complex Variables

Problem 1
Let \( \gamma \) be the unit circle. Then for \( z \in \gamma \), we have by Rouche’s Theorem

\[
|z^8 - 4z^5 + z^2 - 1 + 4z^5| = |z^8 + z^2 - 1| \leq |z^8| + |z^2| + 1 \leq 3 < 4 = |4z^5|
\]

Hence there are 5 zeros with modulus less than one.

Problem 2
Let \( f(z) = (z^2 - z^4)/(1 - z^6) \), and let \( \gamma \) be the contour in the upper half of the plane with radius \( R \). Then by the residue theorem, we have

\[
\int_{\gamma} f(z)\,dz = 2\pi i \sum_{k=1}^{3} \text{Res}(f, z_k)
\]

where \( z_k \) are the three poles in the upper half of the plane. Notice on the outer radius

\[
\left| \int_{|z|=R} f(z)\,dz \right| \leq \frac{2\pi R(R^2 - R^4)}{R^6 - 1} \to 0
\]
as \( R \to \infty \). Thus as \( R \to \infty \)

\[
\int_{-\infty}^{\infty} f(z)\,dz = 2\pi i \left( 0 + \frac{e^{i2\pi/3} - e^{i4\pi/3}}{6e^{i5\pi/3}} + \frac{e^{i4\pi/3} - e^{i8\pi/3}}{6e^{i10\pi/3}} \right)
\]

Problem 3.a
So we can define the square root with a branch cut on the negative real axis. Now for \( z \in [-\pi/2, \pi/2] \) \( f(z) = \sqrt{\cos z} \) is analytic inside the circle \( |z| = \pi/2 \). This can been seen by integrating \( f(z) \) around the circle and see that the integral is zero. Hence By Morera’s Theorem, the function \( f \) is analytic inside the circle. Hence it has a Taylor expansion at \( z = 0 \)

\[
f(z) = \sqrt{\cos z} = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots
\]

Now we know that \( f(z)f(z) = \cos z \). Hence

\[
(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4)(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} \pm \cdots
\]

Hence we know that \( a_0 = 1 \), and \( a_1 = 0 \), \( a_2 = -1/4 \), \( a_3 = 0 \), and \( a_4 = -1/96 \). Hence

\[
\sqrt{\cos z} = 1 - \frac{1}{4} z^2 - \frac{1}{96} z^4 \cdots
\]
Problem 3.b
Clearly the radius of convergence is $\pi/2$, since that’s where we have our branch point.

Problem 4
$$w = z^{\pi/\alpha} - iIm(z_0) + (t - Re(z_0))$$

Problem 5
We have $f$ is entire with zeros at $z_1, z_2, ..., z_n$ with multiplicity $m_1, m_2, ..., m_n$. Recall that $f$ has a zero at $z_1$ with multiplicity $m_1$ if $f^{(m_1)}(z_1) \neq 0$ and $f^{(i)}(z_1) = 0$ for $i < m_1$. So then for

$$g(z) = \begin{cases} 
\frac{f(z)}{(z-z_1)^{m_1}} & z \in \mathbb{C} \setminus z_1 \\
a_{m_1} & z = z_1 
\end{cases}$$

Now notice that $g$ is continuous and entire since

$$\lim_{z \to z_1} g(z) = \lim_{z \to z_1} \frac{f(z)}{(z-z_1)^{m_1}} = \frac{m_1!a_{m_1}}{m_1!} = a_{m_1}$$

where $a_{m_1} \neq 0$. Also $g(z) = 0$ if and only if $f(z) = 0$ with the same multiplicity. By repeating the same argument, we have

$$f(z) = \prod_{i=1}^{n} (z-z_i)^{m_i} h(z)$$

where $h(z)$ is entire and $h(z) \neq 0$ for all $z \in \mathbb{C}$. Hence $1/h(z)$ is entire and $h'(z)$ is entire. Then $h'(z)/h(z)$ is entire. Now we define

$$g(z) = \int_{0}^{z} \frac{h'()}{h()} d\zeta$$

which is entire with $g'(z) = h'(z)/h(z)$. Thus

$$\left(h(z)e^{-g(z)}\right)' = h'(z)e^{-g(z)} - g'(z)h(z)e^{-g(z)} = 0$$

which implies

$$h(z)e^{-g(z)} = c \Rightarrow h(z) = e^{g_2(z)}$$

where $g_2(z)$ is entire. This completes the proof.
Problem 1
We have $A + uv^T$ singular if and only if $A^{-1}(A + uv^T)$ is singular. So notice $A^{-1}(A + uv^T) = I + A^{-1}uv^T$ is singular implies that there exists non-zero vector $x_1$ such that

$$(I + A^{-1}uv^T)x_1 = 0$$

Hence

$$x_1 + A^{-1}uv^Tx_1 = x_1 + (v^Tx_1)A^{-1}u = 0$$

which implies $A^{-1}u = -x_1/(v^Tx_1)$. So notice for $x_2 = x_1/(v^Tx_1)$, $x_2$ is also in the null space. Therefore

$$(I + A^{-1}uv^T)x_2 = 0 = x_2 + (v^Tx_2)A^{-1}u = 0$$

and we must have $A^{-1}u = -x_2$. Therefore $x_2 = -A^{-1}u$ and

$$(I + A^{-1}uv^T)(-A^{-1}u) = 0 \Rightarrow -A^{-1}u - A^{-1}uv^TA^{-1}u = v^TA^{-1}u = -1$$

Problem 2
Recall that if two square matrices $A$ and $B$ are invertible implies that $AB$ is invertible where $(AB)^{-1} = B^{-1}A^{-1}$. Hence $A(A+B)^{-1}B$ and $B(A+B)^{-1}A$ are both invertible and notice

$$A(A+B)^{-1}B = B(A+B)^{-1}A \iff B^{-1}(A+B)A^{-1} = A^{-1}(A+B)B^{-1} \iff B^{-1} + A^{-1} = B^{-1} + A^{-1}$$

which is indeed true.

Problem 3.a
Notice that

$$(Fv,Fw) = \sum_{k=1}^{N} \sum_{p=1}^{N} \sum_{q=1}^{N} \bar{v}_p \bar{w}_q \bar{F}_{k,p} \bar{F}_{k,q}$$

Now notice that

$$\sum_{k=1}^{N} \bar{F}_{k,p} \bar{F}_{k,q} = \sum_{k=1}^{N} \frac{1}{N} e^{2\pi i k(p-q)/N} = 0$$

for $p \neq q$. and

$$\sum_{k=1}^{N} \bar{F}_{k,p} \bar{F}_{k,q} = \sum_{k=1}^{N} \frac{1}{N} e^{2\pi i k(p-q)/N} = 1$$

-50
for \( p = q \). Thus

\[
(Fv, Fw) = \sum_{k=1}^{N} \sum_{p=1}^{N} \sum_{q=1}^{N} v_p w_q F_{k,p} F_{k,q} = \sum_{p=1}^{N} v_p w_p = (v, w)
\]

### Problem 3.b

By part (a) we can see that the inverse of \( F \) is \( F^H \).

### Problem 4.a

Elimination subtracts 2 times row 1 from row 2, likewise 2 from 3, 3 from 4, 4 from 5. Hence

\[
A = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 & 0 \\
0 & 2 & 1 & 0 & 0 \\
0 & 0 & 2 & 1 & 0 \\
0 & 0 & 0 & 2 & 1
\end{pmatrix} = LL^T
\]

### Problem 4.b

Notice we have \( \det(A) = \det(L) \det(L^T) = 1 \)

### Problem 4.c

\( A \) is indeed positive semi-definite. Recall that when a symmetric matrix has one of these four properties, it has all:

**Property 1:** All \( n \) eigenvalues are positive

**Property 2:** All \( n \) upper left determinants are positive

**Property 3:** All \( n \) pivots are positive

**Property 4:** \( x^T A x \) is positive except at \( x = 0 \). Then matrix \( A \) is positive definite.

Hence as we seen above, all pivots of \( A \) were 1’s, and thus positive definite. Or the long way is let \( x = (x_1, x_2, x_3, x_4, x_5) \). Then we have

\[
x^T A x = x_1(x_1 + 2x_2) + x_2(2x_1 + 5x_2 + 2x_3) + x_3(2x_2 + 5x_3 + 2x_4) + x_4(2x_3 + 5x_4 + 2x_5) + x_5(2x_4 + 5x_5)
\]

\[
= (x_1 + 2x_2)^2 + (x_2 + 2x_3)^2 + (x_3 + 2x_4)^2 + (x_4 + 2x_5)^2 + x_5^2
\]

Hence it is clearly positive unless \( x_i = 0 \) for all \( i \). Thus \( A \) is positive-definite.
Problem 5.a

Notice that \[ \left( \begin{array}{c} 1 \\ 5 \end{array} \right) \] is an eigenvector with eigenvalue of 2. Hence

\[ M^4 \left( \begin{array}{c} 1 \\ 5 \end{array} \right) = 2^4 \left( \begin{array}{c} 1 \\ 5 \end{array} \right) \]

Problem 5.b

Notice that the eigenvectors span \( \mathbb{R}^2 \). Hence for any \( v \in \mathbb{R}^2 \) we have

\[ v = c_1 \left( \begin{array}{c} 1 \\ 1 \end{array} \right) + c_2 \left( \begin{array}{c} 1 \\ 5 \end{array} \right) \]

Hence

\[ M^n \left( \begin{array}{c} a \\ b \end{array} \right) = c_1 \left( \begin{array}{c} 1 \\ 3 \end{array} \right)^n \left( \begin{array}{c} a \\ b \end{array} \right) + c_2 2^n \left( \begin{array}{c} a \\ b \end{array} \right) \]

Therefore

\[ \lim_{n \to \infty} M^n \left( \begin{array}{c} a \\ b \end{array} \right) < \infty \]

if \( c_2 = 0 \). Hence all the vectors in the form \( (c_1, c_1) \) for \( c_2 \in \mathbb{R} \) the limit exists.

Jan 1999 Advanced Calculus

Problem 1.a

Notice

\[ \sum_{k=n+1}^{m} u_k v_k = \sum_{k=n+1}^{m} (U_k - U_{k-1}) v_k \]

\[ = (U_{n+1} - U_n) v_{n+1} + (U_{n+2} - U_{n+1}) v_{n+2} + (U_{n+3} - U_{n+2}) v_{n+3} + \cdots + (U_{m-1} - U_{m-2}) v_{m-1} + (U_m - U_{m-1}) v_m \]

\[ = U_m v_m - U_n v_{n+1} + \sum_{k=n+1}^{m-1} U_k (v_k - v_{k+1}) = U_m v_m - U_n v_{n+1} - \sum_{k=n+1}^{m-1} U_k (v_{k+1} - v_k) \]
Problem 1.b

To show that \( \sum a_n(x)b_n(x) \) converges uniformly on \( I \), it suffices to show that we can bound the tail \( |\sum_{n=N}^{\infty} a_n(x)b(x)| < \epsilon \) for all \( \epsilon > 0 \). Now let

\[
s_n(x) = a_1(x) + a_2(x) + \cdots + a_n(x)
\]

then \( a_n(x) = s_n(x) - s_{n-1}(x) \). Since \( b_n(x) \downarrow 0 \) uniformly, and

\[
\sum |b_n(x) - b_{n-1}(x)|
\]

converges uniformly, implies \( \exists N \) such that \( \forall n \geq N \),

\[
|b_n(x)| < \frac{\epsilon}{3M} \quad \text{and} \quad \sum_{n=N+1}^{\infty} |b_n(x) - b_{n-1}(x)| < \frac{\epsilon}{3M}
\]

So

\[
\left| \sum_{n=N+1}^{m} a_n(x)b_n(x) \right| = \left| b_m(x)s_m(x) - b_N(x)s_N(x) + \sum_{k=N+1}^{m-1} s_k(x)(b_{k+1}(x) - b_k(x)) \right|
\]

\[
\leq |b_M(x)s(x)| + |b_n(x)s_{N+1}(x)| + \left| \sum_{k=N+1}^{m-1} s_k(x)(b_{k+1}(x) - b_k(x)) \right|
\]

\[
\leq \frac{\epsilon}{3M} M + \frac{\epsilon}{3M} M + M \sum_{k=N+1}^{m-1} |b_{k+1}(x) - b_k(x)| \leq \epsilon
\]

Since \( |s_n(x)| < M \) for all \( n \). Hence this completes the proof. This is the proof of Abel’s Test:

1: The series \( \sum_{n=1}^{\infty} a_n(x) \) converges uniformly on \( X \)

2: For every \( x \in X \), the sequence \( b_n(x) \) is monotonic

3: There exists \( K \in \mathbb{R} \) such that \( |b_n(x)| \leq K \) for every \( n \in \mathbb{N} \) and every \( x \in X \)

Then

\[
\sum_{n=1}^{\infty} a_n(x)b_n(x)
\]

converges uniformly on \( X \).
Problem 2.a

Notice
\[
\lim_{n \to \infty} \sum_{k=1}^{n} \frac{k}{n^2 + k^2} = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{k/n}{1 + (k/n)^2} = \int_{0}^{1} \frac{x}{1 + x^2} \, dx = \frac{1}{2} \ln 2
\]

Problem 3

Going right to left, notice by performing \(u\) substitution with \(u = \sqrt{y}\)

\[
\frac{1}{\sqrt{\pi}} \int_{0}^{\infty} \frac{y^{-1/2}}{ae^y - 1} \, dy = \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} \frac{2du}{ae^{u^2} - 1} = \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} \frac{\frac{1}{a} e^{-u^2}}{1 - \frac{1}{a} e^{-u^2}} \, du
\]

\[
= \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} \frac{1}{a} e^{-u^2} \sum_{k=0}^{\infty} \frac{1}{a^k} \sum_{k=0}^{\infty} \frac{1}{a^{k+1}} \int_{0}^{\infty} e^{-(k+1)u^2} \, du
\]

Now notice

\[
\int_{0}^{\infty} e^{-(k+1)u^2} \, du = \frac{\sqrt{\pi}}{2(k+1)^{3/2}}
\]

Hence

\[
\sum_{n=1}^{\infty} a^{-n} n^{-1/2}
\]

Problem 5.a

Let \(dl = (dl_1, dl_2, dl_3)\) and \(F = (F_1, F_2, F_3)\). Then

\[
dl \times F = \begin{vmatrix}
dl_1 & dl_2 & dl_3 \\
F_1 & F_2 & F_3
\end{vmatrix} = (dl_2 F_3 - dl_3 F_2, dl_3 F_1 - dl_1 F_3, dl_1 F_2 - dl_2 F_1)
\]

So

\[
\int_{\delta \Sigma} dl \times F = \left( \int_{\delta \Sigma} (0, F_3, -F_2) \cdot dl, \int_{\delta \Sigma} (-F_3, 0, F_1) \cdot dl, \int_{\delta \Sigma} (F_2, -F_1, 0) \cdot dl \right)
\]

By stokes theorem we have

\[
= \left( \int_{\Sigma} \nabla \times (0, F_3, -F_2) \cdot dS, \int_{\Sigma} \nabla \times (-F_3, 0, F_1) \cdot dS, \int_{\Sigma} \nabla \times (F_2, -F_1, 0) \cdot dS \right)
\]

Now for the other side let \(dS = (dS_1, dS_2, dS_3)\) we have

\[
dS \times \nabla = \begin{vmatrix}
dS_1 & dS_2 & dS_3 \\
\frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz}
\end{vmatrix} = \left( \frac{dS_2}{dz} - \frac{dS_3}{dy}, \frac{dS_3}{dx} - \frac{dS_1}{dz}, \frac{dS_1}{dy} - \frac{dS_2}{dx} \right)
\]

-54
So \((dS \times \nabla) \times F = \)
\[
\begin{vmatrix}
  dS_2 \frac{d}{dx} - dS_3 \frac{d}{dy} & dS_3 \frac{d}{dx} - dS_1 \frac{d}{dy} & dS_1 \frac{d}{dx} - dS_2 \frac{d}{dz} \\
  F_1 & F_2 & F_3
\end{vmatrix}
\]
which equals
\[
= \left( dS_3 \frac{dF_3}{dx} - dS_1 \frac{dF_3}{dz} - dS_1 \frac{dF_2}{dy} + dS_2 \frac{dF_2}{dx}, dS_1 \frac{dF_1}{dy} - dS_2 \frac{dF_1}{dx} - dS_2 \frac{dF_3}{dz} + dS_3 \frac{dF_3}{dy} \right) \]

So we have
\[
\int_{\Sigma} (dS \times \nabla) \times F = \int_{\delta \Sigma} dl \times F
\]

**Problem 5.b**

By Stokes Theorem, we have
\[
\int_{\delta \Sigma} u \nabla v \cdot dl = \int_{\Sigma} \nabla \times (u \nabla v) \cdot dS
\]
Now recall that \(\nabla \times (fv) = (\nabla f) \times v + f (\nabla \times v)\). I.e. the curl\((fv) = \text{grad}(f) \times v + f \text{curl}(v)\). (page 1174 Salas). Hence
\[
\int_{\delta \Sigma} u \nabla v \cdot dl = \int_{\Sigma} \nabla \times (u \nabla v) \cdot dS = \int_{\Sigma} [(\nabla u) \times (\nabla v) + u(\nabla \times (\nabla v))] \cdot dS
\]
finally recall that the curl of a gradient is zero, i.e. \(\nabla \times (\nabla f) = 0\). Thus
\[
\int_{\delta \Sigma} u \nabla v \cdot dl = \int_{\Sigma} [(\nabla u) \times (\nabla v) + u(\nabla \times (\nabla v))] \cdot dS = \int_{\Sigma} (\nabla u \times \nabla v) \cdot dS
\]

**Jan 1999 Complex Analysis**

**Problem 1.a**

Let \(g(z) = z^n - f(z)\). Then on the circle \(|z| = 2\), we have
\[
|z^n f(z) - z^n| = |f(z)| < 2^n = |z^n|
\]
which implies \(z^n - f(z)\) has \(n\) roots inside the circle \(|z| = 2\). Also notice on the unit circle, \(|z| = 1\), we have
\[
|z^n - f(z) - z^n| = |f(z)| < 1 = |z^n|
\]
which implies that all \(n\) roots are inside the unit circle according to Rouche’s Theorem. Hence \(z^n - f(z)\) does not vanish inside the annulus \(1 \leq |z| \leq 2\).
Problem 1.b
We can only conclude that there are $n$ zeros inside the unit circle counting multiplicity. It is possible that there are other zeros outside the unit circle. Let $f(z) = (1/2)z^{n+1}$. Then it satisfies all of the above. However there is a zero at $z = 2$.

Problem 2
We first map $D$ to the strip $0 < \text{Im}(z) < 1/2$ by $w = (z-1)/z$. Then we map it to $0 < \text{Im}(v) < \pi$ by $v = 2\pi w$. Finally we map that strip to the upper half plane $U$ by $t = e^{v}$. Hence the mapping is $t = e^{2\pi(z-1)/z}$

Problem 3.a
$\text{Re}(z) = N + 1/2$, implies $z = N + 1/2 + iy$. Hence

$$g(z) = \frac{\cos(\pi(N + 1/2) + i\pi y)}{\sin(\pi(N + 1/2) + i\pi y)} = -i \tanh(\pi y)$$

due to $|g(z)| \leq 1$. Likewise

$$h(z) = \frac{1}{\sin(\pi z)} = \frac{1}{\sin(\pi(N + 1/2) + iy\pi)} = \frac{1}{\sin(\pi(N + 1/2)) \cosh(y\pi)}$$

no $|h(z)| \leq 1$. Therefore both $g$ and $h$ are uniformly bounded.

Problem 3.b
$$\frac{1}{2\pi i} \int \frac{\cos(\pi z)}{\sin(\pi z)(z^2 + 1)} \, dz = \sum_{i} \text{Res}(f, z_i) = \text{Res}(f, i) + \text{Res}(f, -i) + \sum_{k=-N}^{N} \text{Res}(f, k)$$

So

$$\text{Res}(f, i) = \frac{\cos(\pi i)}{\sin(\pi i)(2i)} = \frac{1}{2} \left( e^{-\pi} + e^\pi \right)$$

and

$$\text{Res}(f, -i) = \frac{\cos(-\pi i)}{\sin(-\pi i)(-2i)} = \frac{1}{2} \left( e^{-\pi} + e^\pi \right)$$

and

$$\text{Res}(f, k) = \frac{\cos(k\pi)}{\pi \cos(k\pi)(z^2 + 1) + 2z \sin(\pi z)} = \frac{1}{\pi(k^2 + 1)}$$

Thus

$$\frac{1}{2\pi i} \int \frac{\cos(\pi z)}{\sin(\pi z)(z^2 + 1)} \, dz = \frac{1}{\pi} \sum_{n=-N}^{n} \frac{1}{n^2 + 1} + \frac{e^\pi + e^{-\pi}}{e^{-\pi} - e^\pi}$$

-56
likewise

\[
\frac{1}{2\pi i} \int_{C_N} \frac{1}{\sin(\pi z)(z^2 + 1)} \, dz = \text{Res}(f, i) + \text{Res}(f, -i) + \sum_{n=-N}^{N} \text{Res}(f, n)
\]

So

\[
\text{Res}(f, i) = \frac{1}{\sin(\pi i)(2\pi)} = \frac{1}{e^{-\pi} - e^{\pi}}
\]

and

\[
\text{Res}(f, -i) = \frac{1}{\sin(-\pi i)(-2\pi)} = \frac{1}{e^{\pi} - e^{-\pi}}
\]

and

\[
\text{Res}(f, n) = \frac{1}{\pi \cos(n\pi)(n^2 + 1)} = \frac{(-1)^n}{\pi(n^2 + 1)}
\]

\[
\frac{1}{2\pi i} \int_{C_N} \frac{1}{\sin(\pi z)(z^2 + 1)} \, dz = \frac{2}{e^{-\pi} - e^\pi} + \frac{1}{\pi} \sum_{n=-N}^{N} \frac{(-1)^n}{n^2 + 1}
\]

Problem 3.c

So

\[
\lim_{N \to \infty} \left| \frac{1}{2\pi i} \int_{C_N} \frac{\cos(\pi z)}{\sin(\pi z)(z^2 + 1)} \, dz \right| = \lim_{N \to \infty} \left| \frac{1}{2\pi i} \int_{|z|=N} \frac{\cos(\pi z)}{\sin(\pi z)(z^2 + 1)} \, dz \right| \leq \frac{2\pi R}{2\pi} \frac{M}{R^2 - 1} \to 0
\]
as \(R \to \infty\). Thus

\[
\lim_{N \to \infty} \frac{1}{2\pi i} \int_{C_N} \frac{\cos(\pi z)}{\sin(\pi z)(z^2 + 1)} \, dz = 0 = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{n^2 + 1} + e^\pi + e^{-\pi}
\]

so

\[
\sum_{n=-\infty}^{\infty} \frac{1}{n^2 + 1} = \frac{1}{\pi} \frac{e^\pi + e^{-\pi}}{e^\pi - e^{-\pi}}
\]

Likewise

\[
\lim_{N \to \infty} \left| \frac{1}{2\pi i} \int_{C_N} \frac{1}{\sin(\pi z)(z^2 + 1)} \, dz \right| = \lim_{N \to \infty} \left| \frac{1}{2\pi i} \int_{|z|=R} \frac{1}{\sin(\pi z)(z^2 + 1)} \, dz \right| \leq \frac{2\pi R}{2\pi} \frac{1}{M(R^2 - 1)} \to 0
\]
as \(R \to \infty\). Thus

\[
\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2 + 1} = \frac{2\pi}{e^\pi - e^{-\pi}}
\]
Problem 5.a

Let $I, II, III, IV$ be the first, second, third, and fourth quadrant respectively. Then we know that $if(z)$ is analytic in $I$. Also we know that $if(z)$ is purely real on the real axis, which implies we can apply the reflection principle: $(i(f(\overline{z})))$ is analytic in $IV$. Then notice that by Morera’s theorem, for

$$F_1(z) = \begin{cases} i(f(z)) & z \in I \\ i(f(\overline{z})) & z \in IV \end{cases}$$

$F_1(z)$ is analytic in $I \cup IV$. Now notice $-if(-z)$ is analytic in $III$. We can apply the same reflection principle and Morera’s Theorem, to show that

$$F_2(z) = \begin{cases} -i(f(-z)) & z \in III \\ -i(f(\overline{-z})) & z \in II \end{cases}$$

Now I claim that $F_1 \cup F_2$ is analytic in $|z| > 0$. It suffices to show that

$$\lim_{\epsilon \to 0} i(f(\epsilon + iy)) = \lim_{\epsilon \to 0} -i(f(-\epsilon + iy))$$

which reduces to

$$\lim_{\epsilon \to 0} i(f(\epsilon + iy)) = \lim_{\epsilon \to 0} i(f(\epsilon + iy))$$

But remember $f$ is purely real on the imaginary axis, and hence the equality holds. We also need to show

$$\lim_{\epsilon \to 0} -i(f(-(-\epsilon - iy))) = \lim_{\epsilon \to 0} -i(f(\epsilon + iy))$$

Which implies

$$\lim_{\epsilon \to 0} -i(f(\epsilon + iy)) = \lim_{\epsilon \to 0} -i(f(\epsilon + iy))$$

and since $f$ is purely real on the imaginary axis, the equality holds. This completes the extension for $F = F_1 \cup F_2$.

Problem 5.b

By construction above, $F = F_1 \cup F_2$ is an odd function.

Problem 5.c

Since $|f(z)| \leq |z^{-4}| + |z^{-2}| + 1$ and $f$ is an analytic function in $C \setminus \{0\}$, $f$ has a Laurent expansion at $z = 0$

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

Since $|f(z)| \leq |z^{-4}| + |z^{-2}| + 1$, and $f$ is odd, we have
\[ f(z) = az^{-3} + bz^{-1} \]

where \( |a + b| \leq 3 \).

**Problem 5.d**

Now if \( f \) is purely imaginary on the positive real and imaginary axis, the \( if(z) \) is purely real on the positive real and imaginary axis. Hence by the reflection principle and Morera’s theorem, we can extend \( f \) as an analytic function where

\[
f(z) = \begin{cases} 
if(z) & z \in I \\
-if(z) & z \in IV 
\end{cases}
\]

Then by rotating it, we can define it as an analytic function, in the upper half plane

\[
f(z) = \begin{cases} 
if(iz) & z \in II \\
-if(-iz) & z \in I 
\end{cases}
\]

Then by the reflection principle and Morera’s theorem, we can extend this into a function that is analytic in \( \mathbb{C} \setminus \{0\} \)

\[
f(z) = \begin{cases} 
-if(-iz) & z \in I \\
if(iz) & z \in II \\
-if(-iz) & z \in III \\
if(iz) & z \in IV 
\end{cases}
\]

Hence \( f \) is analytic in \( \mathbb{C} \setminus \{0\} \). Also notice that \( f \) is EVEN! So likewise \( f \) has a Laurent expansion about \( z = 0 \), and by part (c) we have

\[ f(z) = az^{-4} + bz^{-2} + c \]

where \( |a + b + c| \leq 3 \).

**Jan 1999 Linear Algebra**

**Problem 1.1**

\[
M = \begin{pmatrix} 
2 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 
\end{pmatrix}
\]
Problem 1.2

\[ M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

Problem 1.3

\[ M = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

where \( MA \) performs the operation.

Problem 1.4

\[ M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \]

where \( AM \) performs the operation.

Problem 2

Let’s first calculate the eigenvalues. Notice

\[
\det \begin{pmatrix} 1 - \lambda & -1 & 0 \\ -1 & 2 - \lambda & -1 \\ 0 & -1 & 1 - \lambda \end{pmatrix} = (1 - \lambda)((2 - \lambda)(1 - \lambda) - 1) = \lambda(1 - \lambda)(\lambda - 3)
\]

Hence the eigenvalues of \( A \) are 0, 1, and 3. Thus \( A \) is diagonalizable

\[ A = M D M = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}^{-1} \]

Hence

\[ \lim_{n \to \infty} a^n A^n = M \begin{pmatrix} 0 & a^n \\ a^n & 3^n a^n \end{pmatrix} M^{-1} \]

Thus we must have \( a < 1/3 \). 

-60
Problem 3.a

Notice
\[ AP = (I - 2P)P = P - 2P^2 = P - 2P = -P \]

Then notice
\[ A^2 = A(I - 2P) = A + 2P + I \]

Hence \( A^{27} = A \).

Problem 3.b

So \( \det(A^2) = \det(I) = 1 \) implies that \( \det(A) = \pm 1 \). I think the answer is
\[ \det(A) = (-1)^{n-k-1} \]

Problem 3.c

Recall that a square matrix is orthogonal if \( A^T A = I \). Hence notice
\[ (P^T)^2 = (P^2)^T = P^T \]

So

Now I claim that \( 2P^T - 2P + 4P^T P = 0 \). Indeed since notice
\[ 4P^T P = 2P^T + 2P \Rightarrow 4P^T P = 2P^T P + 2P \Rightarrow 4P^T P = 2P^T P + 2P^T P \]

Hence \( A \) is indeed orthogonal.

Problem 4.a

Notice that
\[ U M V^T = 0 \Rightarrow (U^T U)M(V^T V) = 0 \]

and since \( U \) and \( V \) are at full rank, \( U^T U \) and \( V^T V \) are invertible, which implies \( M = 0 \).

Problem 4.b

So
\[ G G^{-1} = (I - U V^T)(I - U W V^T) = I - U W V^T - U V^T + U W V^T U V^T = I \]

implies that
\[ V^T U W = I + W \Rightarrow W V^T U = W + I \Rightarrow (V^T U - I) W = I \]
and $W(V^TU - I) = I$

hence $W = (V^TU - I)^{-1}$.

**Problem 5.a**

We want $U \in \mathbb{R}^{n \times (n-k)}$ such that the columns space the null space of $A$. Now these vectors span the null space if and only if it is orthogonal to all the row vectors of $A$. Now we will use Gram Schmidt. Let $a_i$ be the $i$th row of $A$ and $e_1, ..., e_n$ be the standard basis of $\mathbb{R}^n$. Then we will perform Gram Schmidt on the vectors \{a_1, ..., a_k, e_1, ..., e_n\}. Since $A$ is full rank, we know that the first $k$ vectors will be orthogonalized. Now when we continue with the rest of the vectors $e_1, ..., e_n$, we will throw away $k$ zero vectors. So let $u_1, ..., u_n$ be the output of the process. Then $S(u_1, ..., u_k) = S(a_1, ..., a_k)$ and \{u_{k+1}, ..., u_n\} are all orthogonal to the row space. Hence

$$U = \begin{pmatrix} u_1 & u_2 & \cdots & u_n \end{pmatrix}$$

**Sept 1999 Advanced Calculus**

**Problem 1.a**

Notice we have

$$\frac{\sqrt{n+1} - \sqrt{n-1}}{n} \frac{\sqrt{n+1} + \sqrt{n-1}}{\sqrt{n+1} + \sqrt{n-1}} = \frac{(n+1) - (n-1)}{n(\sqrt{n+1} + \sqrt{n-1})} = \frac{2}{n(\sqrt{n+1} + \sqrt{n-1})} = \frac{2}{n\sqrt{n+1} + \sqrt{n-1}} = \frac{2}{n^{3/2}}$$

Hence

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n-1}}{n} \leq \sum_{n=1}^{\infty} \frac{2}{n^{3/2}} < \infty$$

Hence the sum converges.

**Problem 1.b**

Recall that if $\lim_n n^pu_n = A$, then

(i) $\sum u_n$ converges if $p > 1$ and $A$ is finite.

(ii) $\sum u_n$ diverges if $p \leq 1$ and $A \neq 0$ ($A$ may be infinite)

Therefore apply L’Hospital’s rule, we have

$$\lim_{n \to \infty} n^2 \sin^2(1/n) = \lim_{n \to \infty} \left( \frac{\sin(1/n)}{1/n} \right)^2 = \lim_{n \to \infty} \frac{\sin^2(1/n)}{(1/n)^2} = \lim_{n \to \infty} n \sin(1/n) \cos(1/n)$$

-62
\[ \lim_{n \to \infty} \frac{\cos(1/n)(-n^{-2}) \cos(1/n) - \sin(1/n)(-n^{-2}) \sin(1/n)}{-n^{-2}} = \lim_{n \to \infty} \cos^2(1/n) - \sin^2(1/n) = 1 \]

Hence \( \sum_{n=1}^{\infty} \sin^2(1/n) \) converges.

**Problem 1.c**

Since \( \lim_{n} \sqrt{n} \to 1 \), \( \exists M \) such that \( \sqrt{n} \leq M \) for all \( n \). Hence

\[ \sum_{n=2}^{\infty} \frac{1}{n \sqrt{n} \log n} \geq \sum_{n=2}^{\infty} \frac{1}{nM \log n} \to \infty \]

Hence the summation diverges.

**Problem 2.a**

By using integration by parts, we set

\[ \begin{align*}
  u &= x^{-1/2} \\
  du &= (-1/2)x^{-3/2}dx \\
  v &= -\cos x \\
  dv &= \sin xdx
\end{align*} \]

and we have

\[ \int_{1}^{\infty} \frac{\sin x}{\sqrt{x}} dx = \left[-\cos x \right]_{1}^{\infty} - \frac{1}{2} \int_{1}^{\infty} \frac{\cos x}{x^{3/2}} dx \]

and we know that \( \int_{1}^{\infty} \frac{\cos x}{x^{3/2}} dx \) converges, and hence the original integral converges.

**Problem 2.b**

Notice

\[ \int_{0}^{\infty} \frac{|\sin x|}{\sqrt{x}} dx \geq \sum_{k=0}^{\infty} \int_{k\pi + \pi/6}^{k\pi + 5\pi/6} \frac{|\sin x|}{\sqrt{x}} dx \geq \sum_{k=0}^{\infty} \frac{1}{2} \frac{(5\pi/6 - \pi/6)}{\sqrt{k\pi + 5\pi/6}} = \sum_{k=0}^{\infty} \frac{\pi/3}{\sqrt{\pi(k + 5/6)}} \to \infty \]

or recognize for \( x \geq 0 \), we have \( \sin x \leq x \). Hence

\[ \sum \sin^2(1/n) \leq \sum \left( \frac{1}{n} \right)^n < \infty \]

**Problem 3.a**

Let

\[ f_N(x) = \sum_{n=1}^{N} x^n(1 - x^n) \]
Then clearly $f_N$ converges pointwise since $f(0) = f(1) = 0$ and for $x \in (0, 1)$,

$$\lim_{N \to \infty} f_N(x) = \sum_{n=0}^{\infty} x^n (1 - x^n) \leq \sum_{n=0}^{\infty} x^n = \frac{1}{1 - x} < \infty$$

However it does not converge uniformly since $f$ is not continuous at $x = 1$. Indeed since $f(1) = 0$, but $\forall \epsilon > 0$, we have

$$\sum_{n=1}^{\infty} (1 - \epsilon)^n (1 - (1 - \epsilon)^n) = \sum_{n=1}^{\infty} (1 - \epsilon)^n - \sum_{n=1}^{\infty} (1 - \epsilon)^{2n} = \frac{1}{1 - (1 - \epsilon)} - \frac{1}{1 - (1 - \epsilon)^2} = \frac{1 - \epsilon}{2\epsilon - \epsilon^2} > 1$$

for small enough $\epsilon$.

**Problem 3.b**

Notice

$$\sum_{n=0}^{\infty} (1 - x)^n x^n (1 - x^n) = (1 - x)^\alpha \sum_{n=0}^{\infty} x^n (1 - x^n) = (1 - x)^\alpha x \frac{1}{1 - x} + x \frac{1 - x}{1 + x} = (1 - x)^{\alpha - 1} x$$

Which is not continuous at $x = 1$ if $\alpha < 1$. Now if $\alpha \geq 1$, it is continuous for $x \in [0, 1]$. Since $[0, 1]$ is compact in each function is continuous implies that it converges uniformly.

**Problem 4.a**

Remember the exterior derivative from the workshop

$$\int_{dE} w = \int_{E} dw$$

But anyways, by Greens Theorem we have

$$Area(E) = \int \int_{E} dxdy = \frac{1}{2} \int \int_{E} 1 - (-1)dxdy = \frac{1}{2} \int_{dE} xdy - ydx$$

**Problem 4.b**

By the divergence theorem, we have for $v = (1/2)(x, y, z)$

$$Vol(E) = \int \int \int_{V} dxdydz = \int \int \int_{V} \nabla \cdot v dxdydz = \int \int_{dE} v \cdot ndS$$

$$= \frac{1}{3} \int \int_{dE} (x, y, z) \cdot ndS = \frac{1}{3} \int \int_{dE} xdydz + ydzdx + zdxdy$$
Problem 4.c

The best way to solve this is

\[ \int_{x=0}^{1} \int_{y=0}^{-x+1} z \, dy \, dx = \int_{x=0}^{1} \int_{y=0}^{-x+1} (1-x-y) \, dy \, dx = \int_{x=0}^{1} \left( \frac{1}{2} x^2 - x + \frac{1}{2} \right) \, dx = \frac{1}{6} \]

Problem 5

We have \( f = x^2 + y^2 + z^2 \) and \( g_1 = x + y - z = 0 \) and \( g_2 = x^2/4 + y^2/5 + z^2/25 = 1 \). Thus

\[ dT = \begin{pmatrix} 2x & 2y & 2z \\ 1 & 1 & -1 \\ x/2 & 2y/5 & 2z/25 \end{pmatrix} \]

Now we must have \( \det(dT) = 0 \). Hence this implies

\[ 16yz = 21xz + 5xy \]

Now by \( g_1 \) we have \( z = x + y \), and so

\[ 16y(x+y) = 21x(x+y) + 5xy \Rightarrow 0 = (7x + 8y)(3x - 2y) \]

So one set of possible solutions is when \( x = -8y/7 \). Then by using \( g_2 \) we have

\[ \frac{16y^2}{49} + \frac{y^2}{5} + \frac{y^2}{49(25)} = 1 \]

which implies

\[ y = \pm \frac{35}{\sqrt{646}} \]

Hence our first set of solutions is

\[ x = -\frac{8}{7} \left( \frac{35}{\sqrt{646}} \right), y = \frac{35}{\sqrt{646}}, z = -\frac{1}{7} \frac{35}{\sqrt{646}} \]

and our second is

\[ x = \frac{8}{7} \left( \frac{35}{\sqrt{646}} \right), y = -\frac{35}{\sqrt{646}}, z = \frac{1}{7} \frac{35}{\sqrt{646}} \]

Now if \( x = 2y/3 \) we have

\[ \frac{4y^2}{9(4)} + \frac{y^2}{5} + \frac{(2y/3 + y)^2}{25} = 1 \]

which implies

\[ y = \pm \frac{\sqrt{45}}{19} \]
So our third set of solutions is

\[ x = \frac{2}{3} \sqrt{\frac{45}{19}}, y = \sqrt{\frac{45}{19}}, z = \frac{5}{3} \sqrt{\frac{45}{19}} \]

and our fourth set is

\[ x = \frac{-2}{3} \sqrt{\frac{45}{19}}, y = \sqrt{-\frac{45}{19}}, z = \frac{5}{3} \sqrt{-\frac{45}{19}} \]

And by direct calculation, we can see that the first and second set of solutions is minimum, and the third and fourth is the max.

**Sept 1999 Complex Variables**

**Problem 1**

Remember from the workshop that when we integrate through a simple pole at \( z_0 \), then the integral over \( C_\epsilon \) is just

\[ \frac{1}{2} 2\pi i \text{Res}(f, z_0) \]

Very useful. But anyways, let \( \gamma \) be the contour of two upper semi-circles with radius \( \epsilon \) and \( R \). Then let \( f(z) = \frac{e^{iz}}{z} \), and we have

\[ \int_\gamma f(z)dz = 0 \]

Now notice

\[ \int_\gamma f(z)dz = \int_{-R}^{-\epsilon} \frac{e^{iz}}{z}dz + \int_{C_\epsilon} \frac{e^{iz}}{z}dz + \int_{\epsilon}^{R} \frac{e^{iz}}{z}dz + \int_{C_R} \frac{e^{iz}}{z}dz = 0 \]

Thus

\[ \int_\gamma f(z)dz = \int_{R}^{\epsilon} \frac{e^{iz} - e^{-iz}}{z}dz + \int_{C_\epsilon} \frac{e^{iz}}{z}dz + \int_{C_R} \frac{e^{iz}}{z}dz = 0 \]

which implies

\[ \int_{R}^{\epsilon} \frac{e^{iz} - e^{-iz}}{z}dz = - \int_{C_\epsilon} \frac{e^{iz}}{z}dz - \int_{C_R} \frac{e^{iz}}{z}dz = 0 \]

On \( C_R \) we have \( z = Re^{i\theta} \), and we have

\[ \left| \int_{C_R} \frac{e^{iz}}{z}dz \right| = \int_0^{\pi} \frac{e^{iR(\cos \theta + \sin \theta)}}{R}Rie^{i\theta}d\theta \leq \int_0^{\pi} e^{-R\sin \theta}d\theta = 2 \int_0^{\pi/2} e^{-R\sin \theta}d\theta \]

Now recall that \( \sin \theta \geq 2\theta/\pi \) for \( 0 \leq \theta \leq \pi/2 \). Hence
\[
2 \int_0^{\pi/2} e^{-R \sin \theta} d\theta \leq 2 \int_0^{\pi/2} e^{-R \theta / \pi} d\theta = \frac{\pi}{R} (1 - e^{-R}) \to 0
\]
as \(R \to \infty\). Now notice that on \(C_\epsilon\) we have \(z = \epsilon e^{i\theta}\) and

\[
\lim_{\epsilon \to 0} \int_{C_\epsilon} \frac{e^{iz}}{z} \, dz = \lim_{\epsilon \to 0} \int_0^{\pi} \frac{e^{i\epsilon e^{i\theta}}}{\epsilon e^{i\theta}} i\epsilon e^{i\theta} \, d\theta = \lim_{\epsilon \to 0} \int_0^{\pi} i\epsilon e^{i\theta} \, d\theta = -\int_0^{\pi} ie^{i\epsilon e^{i\theta}} \, d\theta = -i\pi
\]

There as \(\epsilon \to 0\) and \(R \to \infty\) we have

\[
\int_\epsilon^{\infty} \frac{e^{iz} - e^{-iz}}{z} \, dz = 2i \int_0^{\infty} \frac{\sin z}{z} \, dz = i\pi
\]

Hence we have

\[
\int_0^{\infty} \frac{\sin z}{z} \, dz = \frac{\pi}{2}
\]

Since \(\frac{\sin z}{z}\) is an even function, we have

\[
\int_{-\infty}^{\infty} \frac{\sin z}{z} \, dz = \pi
\]
as long as \(\sin x/x = 1\) when \(x = 0\).

**Problem 2.a**

Yes a function exists and it is NOT unique. Notice

\[
F(z) = \frac{(z-1)(z-2)(z-3)}{(z-4)(z-5)} e^{g(z)}
\]

where \(g\) is an entire function. So

\[
F(0) = 1 = \left(\frac{-3}{10}\right) e^{g(0)} \Rightarrow \frac{-10}{3} e^{x+iy}
\]

Hence \(x = \ln(10/3)\) and \(y = (2n+1)\pi\) for \(n \in \mathbb{Z}\). But it’s not unique since we can have

\[
F(z) = \frac{(z-1)(z-2)(z-3)}{(z-4)(z-5)} e^{\ln(10/3)+i\pi+x}
\]
or

\[
F(z) = \frac{(z-1)(z-2)(z-3)}{(z-4)(z-5)} e^{\ln(10/3)+i\pi}
\]

would work.
Problem 2.b

Now in order to have polynomial growth at $\infty$, we need $e^g(z)$ to have polynomial growth at $\infty$. Notice

$$g(x) \leq M \log |z|$$

by Cauchy estimates and since $g$ is entire, implies that $g$ must be constant. Hence $g$ must be constant and we must have

$$F(z) = \frac{(z-1)(z-2)(z-3) -10}{3(z-4)(z-5)}$$

Problem 2.c

If $F$ is bounded at $\infty$, then we must have $e^g(z)$ at least bounded. But then we must have

$$F(z) = \frac{(z-1)(z-2)(z-3) -10}{3(z-4)(z-5)}$$

Hence no function exists since the other part goes to infinity.

Problem 3

Now we map the first quadrant to the UHP by $w = z^2$. Then we map the UHP to the unit disk by linear transformation

$$w_2 = \frac{aw+b}{cw+d} = \frac{a'w+b'}{w+d'}$$

Now since we must satisfy $0 \mapsto i$ and $\infty \mapsto -i$, we have $a' = -i$ and $b' = id'$. Hence let $b' = -1$ and $d' = i$. Then we have

$$w_2 = \frac{-iw-1}{w+i} = \frac{-iz^2 - 1}{z^2 + i}$$

Now the mapping is not unique, since we could have let $b' = -2$ and $d' = 2i$. Then

$$w_2 = \frac{-iw-2}{w+2i} = \frac{-iz^2 - 2}{z^2 + 2i}$$

which does the same conformal mapping.

Problem 4

Lets define the branch cut for $\sqrt{\cdot}$ on the negative real axis. Notice for $z \in (-\pi,0) \subset R$, $\sin(z)$ and $z$ are both negative. Hence we can define

$$f(z) = \sqrt{z} \sin z$$

to be $f(0) = 0$ and continuous in $|z| < \pi$. Then we just will just fill in values for $f$ on the negative real axis to make $f$ continuous. This is possible since both $z$ and $\sin z$ are negative real values from
(-\pi, 0). Then notice the integral around the contour |z| = \pi is equivalent to the integral around two semicircles in the upper and lower half planes, only \epsilon above the real line. Hence as \epsilon \to 0, the integrals are equivalent since f is continuous on the real line. So by Cauchy, we know that the integral around both semicircles is 0, and so the integral of f on |z| = \pi is zero. Therefore my Morera’s Theorem, f is analytic in |z| < \pi.

Now for the second part, I claim that the radius of convergence is \pi. By above we know it is at least \pi. Notice if z > \pi, on the real line, z is positive and \sin z is negative, and so f is not continuous as we cross the x-axis. Therefore it is not analytic and the radius of convergence is \pi.

Problem 5

For y > 0,

\[
f(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \sum_j e^{ia_j x + ia_j t} \frac{dt}{t^2 + y^2} = \frac{y}{\pi} \sum_j c_j e^{ia_j x} \int_{-\infty}^{\infty} \frac{e^{ia_j t}}{t^2 + y^2} dt.
\]

now notice for

\[
\int_{-\infty}^{\infty} \frac{e^{ia_j t}}{t^2 + y^2} dt
\]

Let \gamma be the contour of the semi-circle with radius R in the upper-half plane. Then

\[
\left| \int_{C_R} \frac{e^{ia_j z}}{z^2 + y^2} dz \right| \leq \pi R \frac{\max\{e^{-Ra_j \sin \theta}\}}{R^2 - y^2} \leq \pi R \frac{\max\{e^{-a_j \pi / 2}\}}{R^2 - y^2} = \frac{\pi R}{R^2 - y^2} \to 0
\]

as Hence by the residue theorem

\[
\int_{-\infty}^{\infty} \frac{e^{ia_j t}}{t^2 + y^2} dt = 2\pi i \frac{e^{-a_j y}}{2iy} = \frac{\pi}{y} e^{-a_j y}
\]

Hence let

\[
g(x, y) = \sum_j c_j e^{ia_j x - a_j y} = \sum_j c_j e^{ia_j z}
\]

Which everyone knows is analytic in the upper and lower half planes. We also know that the imaginary and real part of an analytic function is harmonic. Therefore

\[u(x, y) = \text{Im}(g(x, y))\]

is indeed harmonic in the upper and lower half planes. Also notice that the function is entire, which implies that in general the function defined by u(x, y) for y \neq 0 and by f(x) for y = 0 is harmonic.

Sept 1999 Linear Algebra

Problem 1.a

So we have

-69
\[
x^2 \frac{d^2}{dx^2} = \begin{pmatrix}
0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 2 & 0 & \cdots & 0 \\
0 & 0 & 0 & 2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & n(n-1)
\end{pmatrix}
\]
and
\[
-bx \frac{d}{dx} = -b \begin{pmatrix}
0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 2 & 0 & \cdots & 0 \\
0 & 0 & 0 & 3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & n
\end{pmatrix}
\]
\[
-c \frac{d}{dx} = -c \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & n \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}
\]
Hence
\[
A = \begin{pmatrix}
0 & -c & 0 & 0 & \cdots & 0 \\
0 & -b & -2c & 0 & \cdots & 0 \\
0 & 0 & 2 - 2b & -3c & \cdots & 0 \\
0 & 0 & 0 & 2 \cdot 3 - 3b & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & n(n-1) - nb
\end{pmatrix}
\]
therefore the eigenvalues are \(\{0, -b, 2 - 2b, 2 \cdot 3 - 3b, \ldots, (n-1)n - nb\}\).

**Problem 1.b**

We know that we can diagonalize \(A\) if \(A\) has all distinct eigenvalues. Hence if \(c \in \mathbb{R}\) and \(b \neq 0, 1, 2, \ldots, n - 1\), we can diagonalize \(A\). Now if \(b \in \{0, 1, \ldots, n - 1\}\), then \(A\) has \(n - 1\) distinct eigenvalues and \(\lambda_i = 0\) twice. So if \(c = 0\), then the null space of \(A\) has dimension 2 and so the eigenvectors for \(\lambda_i = 0\) are linearly independent. Therefore we can diagonalize \(A\). Now if \(c \neq 0\) notice
\[
\mathbb{R}(A) = n \Rightarrow n(A) = 1
\]
Hence there is only one linearly independent eigenvector of \(\lambda_i = 0\). Hence it is not diagonalizable.
Problem 2.a
They are \( p(x) = (x - 2)(x - 1)^2 \) and \( p(x) = (x - 2)^2(x - 1) \).

Problem 2.b
Notice \( A^2 - 3A + 2I = 0 \) implies that for all \( x \in \mathbb{R}^3 \), we have
\[
(A - 2I)(A - I)x = 0
\]
So the only possible eigenvalues are 2 and 1. for \( \lambda = 1 \) we cannot have a Jordan Block of size > 1. indeed notice
\[
Av_2 = v_1 + v_2 \Rightarrow A^2v_2 = v_1 + Av_2 \Rightarrow -v_1 = 0
\]
and hence we have a contradiction. Likewise for \( \lambda = 2 \) we cannot have a Jordan Block of size > 1. Indeed that would imply
\[
Av_2 = v_1 + 2v_2 \Rightarrow A^2v_2 = Av_1 + 2Av_2 = 4v_1 + 4v_2 \Rightarrow v_1 = 0
\]
Thus the only Jordan forms we can have are
\[
J_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad J_2 = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}
\]
\[
J_3 = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} \quad J_4 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}
\]

Problem 3
By Gaussian elimination we have
\[
\begin{array}{ccc|c}
  c & b & c & 1 + b - c \\
  b & 1 & b & b \\
  c & b & c & 1 - b + c \\
\hline
  c & b & c & 1 + b - c \\
  0 & -b^2/c + 1 & 0 & -b/c - b^2/c + 2b \\
  0 & 0 & 0 & -2b + 2c \\
\end{array}
\]
So we have no solutions if \( b \neq c \), an infinite number of solutions if \( b = c \). So notice we cannot have a unique solution to our system.
Problem 4

Let’s first solve for the eigenvalues. Notice

\[
\det(A - I\lambda) = \det \begin{pmatrix} -1 - \lambda & 1 & 1 \\ 1 & -1 - \lambda & 1 \\ 1 & 1 & -1 - \lambda \end{pmatrix} = -\lambda^3 - 3\lambda^2 + 4
\]

Then we can see that \(\lambda = 1\) is a solution. Thus by factoring, we have

\[-\lambda^3 - 3\lambda^2 + 4 = -(x - 1)(x + 2)^2\]

Now notice that the eigenvectors of \(\lambda = -2\) are linearly independent since \((-1, 0, 1)^T\) and \((-1, 1, 0)^T\) would work. Hence \(A\) is diagonalizable

\[
A = MDM^{-1} = \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix} -2 & \& \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix} | & | & | \\ -2 & 1 \\ 1 \end{pmatrix} \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \end{pmatrix} \end{pmatrix}^{-1}
\]

which implies for all integer \(n\)

\[
A^n = \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix} -2^n & \& \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix} | & | & | \\ -2^n & 1 \\ 1 \end{pmatrix} \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \end{pmatrix} \end{pmatrix}^{-1}
\]

and \(v_1, v_2\) are eigenvectors corresponding to eigenvalue \(-2\) and \(v_3\) is the eigenvector corresponding to \(1\).

Jan 2000 Advanced Calculus

Problem 1.a

So we have

\[
f(x) = \sum_{n=1}^{\infty} \frac{\sqrt{n} + x}{1 + n^2 x}
\]

for \(x > 0\), notice

\[
f(x) \leq \sum_{n=1}^{\infty} \frac{\sqrt{n} + x}{n^2 x} = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 x} + \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty
\]

by the comparison test. We can also see that when \(x = 0\), the series diverges. Now let’s look at when \(x < 0\). Clearly if \(\sqrt{\frac{1}{|x|}} \in \mathbb{N}\), then the series diverges. Thus let \(S = \{-1, -1/2, -1/4, -1/9, -1/16/...\} \cup \{0\}\). Thus for \(x < 0\) and \(x \notin S\), we have

\[
\sum_{n=1}^{\infty} \left| \frac{\sqrt{n} + x}{1 + n^2 x} \right| \leq \sum_{n=1}^{\infty} \frac{\sqrt{n} + |x|}{n^2 |x|} - 1 \leq M \sum_{n=1}^{\infty} \frac{\sqrt{n} + |x|}{n^2 |x|} < \infty
\]

by above. Hence it converges absolutely which implies that it converges. Hence \(f(x)\) converges for \(x \in \mathbb{R} \setminus S\).
Problem 1.b
On \( x \in \mathbb{R} \setminus S \), we need to show for all \( \epsilon > 0 \) \( \exists \delta \) such that if \( (x - x_0) < \delta \) which implies

\[
|f(x) - f(x_0)| < \epsilon
\]

indeed notice

\[
|f(x) - f(x_0)| = \left| \sum_{n=1}^{\infty} \frac{\sqrt{n} + x}{1 + n^2 x} - \sum_{n=1}^{\infty} \frac{\sqrt{n} + x_0}{1 + n^2 x_0} \right|
\]

\[
= \left| \sum_{n=1}^{\infty} \frac{(x - x_0) + n^{2.5}(x_0 - x)}{(1 + n^2 x)(1 + n^2 x_0)} \right| = \delta \left| \sum_{n=1}^{\infty} \frac{n^{2.5} - 1}{(1 + n^2 x)(1 + n^2 x_0)} \right|
\]

So let \( \delta = \epsilon/M \). Then we have \( < \epsilon \). So \( f \) is indeed continuous.

Problem 2.a
Notice we have

\[
\frac{df}{dx} = \frac{(3x^2 - y^3)(x^2 + y^2) - 2x(x^3y - xy^3)}{(x^2 + y^2)^2} = \frac{y((x^2 + y^2)^2 + 2y^2(x^2 - y^2))}{(x^2 + y^2)^2}
\]

Hence it is continuous everywhere except possibly at \((0,0)\). So let’s check using polar coordinates.

\[
\frac{r \sin \theta (r^4 + 2r^2 \sin^2 \theta (r^2 \cos(2\theta)))}{r^4} = r(1 + 2 \sin^2 \theta \cos(2\theta))r \to 0
\]

as \( r \to 0 \). Hence regardless of what direction (regardless of \( \theta \)), \( \lim_{(x,y) \to (0,0)} \frac{df}{dx} = 0 \). Hence it is continuous. Similar argument for \( \frac{df}{dy} \).

Problem 2.b
Now we know that \( \frac{d^2}{dx dy} = \frac{d^2}{dy dx} \). So we have

\[
\frac{d}{dy} \frac{y((x^2 + y^2)^2 + 2y^2(x^2 - y^2))}{(x^2 + y^2)^3} = \frac{(x^4 + 12x^2y^2 - 5y^4)(x^2 + y^2) - 4y(x^4y + 4x^2y^3 - y^5)}{(x^2 + y^2)^3}
\]

Hence it exists at every point in \( \mathbb{R}^2 \), except possibly at \((0,0)\). Notice when we approach \((0,0)\) on the \( x \)-axis, we have \( x \to 0 \) and \( y = 0 \). Hence

\[
\lim_{\text{stackrel{x \to 0}{y=0}}} \frac{(x^4 + 12x^2y^2 - 5y^4)(x^2 + y^2) - 4y(x^4y + 4x^2y^3 - y^5)}{(x^2 + y^2)^3} = \lim_{x \to 0} \frac{x^6}{x^6} = \lim_{x \to 0} 1 = 1
\]

However when we approach \((0,0)\) on the \( y = x \) line, we have

\[
\lim_{\text{stackrel{x \to 0}{y=x}}} \frac{(x^4 + 12x^2y^2 - 5y^4)(x^2 + y^2) - 4y(x^4y + 4x^2y^3 - y^5)}{(x^2 + y^2)^3} = \lim_{x \to 0} \frac{16x^6 - 16x^6}{8x^6} = 0
\]

Hence the limit does not exist at \((0,0)\).
Problem 2.c

Clearly

\[ \frac{d^2f}{dydx}(0,0) = \frac{d^2f}{dxdy}(0,0) = 0 \]

Problem 3.a

Notice

\[ f(x) = 2 \sin(\pi x) = 4 \sin(\pi x/2) \cos(\pi x/2) = -4 \sin(\pi x/2) \sin(\pi(x-1)/2) \]

So let

\[ P_n(x) = -4 \left( \frac{\pi x/2}{3!} - \frac{(\pi x/2)^3}{5!} + \cdots + (-1)^{n+1} \frac{(\pi x/2)^{2n+1}}{(2n+1)!} \right) \]

\times \left( \frac{\pi/2(x-1)}{3!} - \frac{(\pi/2(x-1))^3}{5!} + \cdots + (-1)^{n+1} \frac{(\pi/2(x-1))^{2n+1}}{(2n+1)!} \right) \]

Then we have \( P_n(0) = P_n(1) = 0 \) and \( P_n \to f \) uniformly since

\[ \left( \frac{\pi x/2}{3!} - \frac{(\pi x/2)^3}{5!} + \cdots + (-1)^{n+1} \frac{(\pi x/2)^{2n+1}}{(2n+1)!} \right) \xrightarrow{\text{uniformly}} \sin(\pi x/2) \]

\[ \left( \frac{\pi/2(x-1)}{3!} - \frac{(\pi/2(x-1))^3}{5!} + \cdots + (-1)^{n+1} \frac{(\pi/2(x-1))^{2n+1}}{(2n+1)!} \right) \xrightarrow{\text{uniformly}} \sin(\pi(x-1)/2) \]

Problem 4.a

Notice by the divergence theorem, we have

\[ \int \int_S x \cdot ndS = \int \int \int \nabla \cdot x dV = \int \int \int_\Omega 3dV = 3(\text{Area of ellipsoid}) \]

So what is the area of the ellipsoid. Recall that the area of an ellipse

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \]

is \( \pi a \cdot b \). This can be shown by integrating the ellipse using polar coordinates \( r = a \cos \theta + b \sin \theta \). So for the ellipsoid, each \( z \) value gives us a slice of the ellipsoid, which is an ellipse

\[ \frac{x^2}{4(1-z^2/25)} + \frac{y^2}{9(1-z^2/25)} = 1 \]

which has an area of \( \pi/2(13-13z^2/25) \). So the area of the ellipsoid is

\[ \int_{z=-5}^{5} \frac{\pi/2(13-13z^2/25)}{dz} = \pi \left( 13 \cdot 5 - \frac{13 \cdot 125}{75} \right) = \frac{130\pi}{3} \]
Problem 4.b
So we have

\[ P(X) = (x, y, z) \cdot n(X) \]

where

\[ n(X) = \frac{(2x/4, 2y/9, 2z/25)}{\sqrt{\frac{4x^2}{16} + \frac{4y^2}{9} + \frac{4z^2}{25}}} \]

So

\[ P(X) = \frac{2}{\sqrt{\frac{4x^2}{16} + \frac{4y^2}{9} + \frac{4z^2}{25}}} \]

Notice the denominator is just the distance. So Max Point is \((2, 0, 0)\) and Min point is \((0, 0, 5)\).

\[ P_{\text{max}} = 2 \quad P_{\text{min}} = \frac{10}{2} \]

Jan 2000 Complex Variables
Problem 1
Since

\[ \sinh^2 z = \left( \frac{e^z - e^{-z}}{2} \right)^2 = \frac{e^{2z} - 2 + e^{-2z}}{4} = \frac{1}{2} \left( \frac{(2z)^2}{2!} + \frac{(2z)^4}{4!} + \frac{(2z)^6}{6!} + \cdots \right) \]

we have

\[ \oint_C \frac{\cosh z}{z \sinh^2 z} dz = \oint_C \frac{1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots}{z^3 \left( \frac{z^2}{2!} + \frac{2z^4}{4!} + \frac{2z^6}{6!} + \cdots \right)} dz \]

Hence the only pole is at \(z = 0\). By the residue theorem, we have

\[ \oint_C \frac{\cosh z}{z \sinh^2 z} dz = 2\pi i \text{Res} \ (f, 0) \]

So notice

\[ \frac{\cosh z}{z \sinh^2 z} = 2 \frac{1 + \frac{z^2}{2!} + \frac{4 z^4}{4!} + \cdots}{z^3 \left( \frac{2^2}{2!} + \frac{2^4 z^2}{4!} + \frac{2^6 z^4}{6!} + \cdots \right)} = a_{-3} z^{-3} + a_{-2} z^{-2} + a_{-1} z^{-1} + \cdots \]

which implies

\[ 2 \left( 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots \right) = z^3 \left( \frac{2^2}{2!} + \frac{2^4 z^2}{4!} + \frac{2^6 z^4}{6!} + \cdots \right) \left( a_{-3} z^{-3} + a_{-2} z^{-2} + a_{-1} z^{-1} + \cdots \right) \]
so we have $a_{-3} = 1, a_{-2} = 0, a_{-1} = 1/6$. Thus

$$\oint_{C} \frac{\cosh z}{z \sinh^2 z} \, dz = 2\pi i (1/6) = \frac{\pi i}{3}$$

**Problem 2.a**

Using Rouche’s Theorem, we have for $|z| = 1$

$$|z^5 - 12z^2 + 14 - 14| = |z^5 - 12z^2| \leq |z^5| + |12z^2| = 13 < 14 = |14|$$

Thus there are no zeros inside the unit circle.

**Problem 2.b**

Using Rouche’s Theorem, we have for $|z| = 2$

$$|z^5 - 12z^2 + 14 + 12z^2| = |z^5 + 14| \leq |z^5| + 14 = 46 < 48 = |12z^2|$$

Hence there are two zeros inside the annulus $1 \leq |z| < 2$.

**Problem 2.c**

Using Rouche’s Theorem, we have for $|z| = 5/2$

$$|z^5 - 12z^2 + 14 - z^5| = |-12z^2 + 14| \leq |12z^2| + 14 = 89 < 97.6563 = |z^5|$$

Thus there are 3 zeros inside the annulus $2 \leq |z| < 5/2$.

**Problem 3.a**

We have $z = x + iy$ where $x^2 + y^2 = 1$. Thus

$$|g(z)| = \left| \frac{(x - a) + i(y - b)}{(1 - ax - by) + i(by - ay)} \right| = \frac{1 - 2ax - 2y + a^2 + b^2}{1 - 2ax - 2y + a^2 + b^2} = 1$$

and

$$g'(z) = \frac{(1 - cz) + c(z - c)}{(1 - cz)^2} = 1 - |c|^2$$

**Problem 3.b**

The answer is

$$f(z) = \prod_{i=1}^{N} \left( z - \frac{z_i}{1 - z_i z} \right) e^{h(z)}$$

where $h(z)$ is analytic on $|z| \leq 1$. Notice
$|Re(h(z))| = 0$
on $|z| = 1$. By the Max Modulus theorem for the real part, we know that $|Re(h(z))| \leq 0$ inside $|z| \leq 0$. Thus $Re(h(z)) = 0$ inside $|z| \leq 1$. Hence since $h$ is analytic inside $|z| \leq 1$ and $Re(h(z))$ is constant, implies that $h$ is constant inside $|z| \leq 1$. Therefore

$$f(z) = \left( \prod_{i=1}^{N} \frac{z - z_i}{1 - \overline{z_i}z} \right) e^{i\alpha}$$

for $\alpha \in \mathbb{R}$.

**Problem 3.c**

$f$ is analytic on $|z| \leq 1$. So we have

$$f(z) = \left( \frac{z - (1/2 + i/2)}{1 - (1/2 - i/2)z} \right)^3 e^{i\alpha}$$

which implies

$$f'(0) = -3ie^{i\alpha} = 3$$

and so $e^{i\alpha} = i$. Hence

$$f(z) = i \left( \frac{z - (1/2 + i/2)}{1 - (1/2 - i/2)z} \right)^3$$

and so it is unique.

**Problem 4**

Here we can see that $w$ maps $D$ to the lower half plane, where $A' = \sqrt{2}/2, B' = -\sqrt{2}/2, C' = \infty$. The Branch cut $C$ to $\infty$ is $Im(w) = 0$ and $Re(w) \geq 1$ since $z = x + iy$ and $y = 0, x \geq 1$. we have

$$x \quad \sqrt{x^2 - 1}$$

which is a real number since $x \geq 1$ and ranges from 1 to $\infty$. Now the segment $OC$ maps to $Re(w) = 0$ and $Im(w) \leq 0$ since $z = x + iy$ and $y = 0$ and $0 \leq x \leq 1$. Thus

$$w = \frac{x}{\sqrt{x^2 - 1}}$$

is a purely imaginary number since $x \leq 1$. It ranges from $-\infty$ to 0.
Problem 5

The roots for the denominator are $z = r \pm \sqrt{r^2 - 1}$. Hence the radius of convergence is $r - \sqrt{r^2 - 1}$ (or the other root, doesn’t matter). Now clearly the function is analytic at $z = 0$. Hence $\exists \epsilon > 0$ such that for all $z$ such that $|z| < \epsilon$, $f(z)$ is continuous. Hence $f(z)$ has a Taylor series, and

$$f(z) = 1 + A_1(r)z + A_2(r)z^2 + \cdots$$

Now by the Cauchy integral formula and the Residue Theorem, we have

$$A_n(r) = \frac{1}{2\pi i} \int_{|z| = \epsilon} \frac{f(\zeta)}{\zeta^{n+1}} d\zeta = \frac{1}{n!} \frac{d^n}{dz^n} \left[(1 - 2rz + z^2)^{-1/2}\right]_{z=0}$$

Hence we can see that

$$A_n(r) = \frac{1}{n!} \prod_{k=1}^{n} \frac{(2k - 1)}{2^n} r^n + O(r^{n-1})$$

Hence $A_n(r)$ are polynomials of degree $n$.

Jan 2000 Linear Algebra

Problem 1

For $p(x) = a_0 + a_1x + a_2x^2$, we have

$$L(p) = \begin{pmatrix} a_0 \\ a_0 + a_1 + a_2 \\ a_0 + 2a_2 + 4a_2 \\ a_0 + 3a_1 + 9a_2 \end{pmatrix}$$

Hence the kernel is the trivial space. The image of $L$ is $S((1, 1, 1, 1), (0, 1, 2, 3), (0, 1, 4, 9))$. Finally in order to find $im(L)^\perp$, we need to solve the equation

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0$$

Hence we have $x_4 = t$, $x_3 = -3t$, $x_2 = 3t$, and $x_1 = -t$. So $im(L)^\perp = S((-1, 3, -3, 1))$.

Problem 2

Let $f = p(x)e^{3x}$, where $p$ is a polynomial of degree at $n$. Then

$$L(f) = p''(x)e^{3x}$$

Hence the basis of the kernel is $\{e^{3x}, xe^{3x}\}$ and the basis for the image is $\{e^{3x}, xe^{3x}, x^2e^{3x}, \ldots, x^{n-2}e^{3x}\}$. Our transformation is essentially $D^2$. 

-78
\[
D^2 = \begin{pmatrix}
0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 12 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & n(n-1) \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

So the eigenvalues of this transformation is 0. Now for \( n = 3 \), we have

\[
L = \begin{pmatrix}
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 6 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

**Problem 3**

By Gaussian elimination, we have

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>5</td>
<td>6</td>
<td>11</td>
</tr>
<tr>
<td>7</td>
<td>8</td>
<td>a</td>
<td>b</td>
</tr>
</tbody>
</table>

\[
\begin{array}{ccc|c}
1 & 2 & 3 & 10 \\
0 & -3 & -6 & -29 \\
0 & -6 & -21+a & -70+b \\
1 & 2 & 3 & 10 \\
0 & 3 & 6 & 29 \\
0 & 0 & a-9 & -12+b \\
\end{array}
\]

Hence the system has a unique solutions when \( a \neq 9 \) and \( b \in \mathbb{R} \). The system has no solutions when \( a = 9 \) and \( b \neq 12 \). Finally the system has infinite number of solutions when \( a = 9 \) and \( b = 12 \).

**Problem 4.a**

We know that \( A \) has \( m \) distinct eigenvalues, which implies that \( A \) is diagonalizable. Hence

\[
A = MDM^{-1} = \begin{pmatrix}
| & | & \cdots & | \\
v_1 & v_2 & \cdots & v_m \\
| & | & \cdots & | 
\end{pmatrix}
\begin{pmatrix}
\lambda_1 \\
\lambda_2 \\
\vdots \\
\lambda_m
\end{pmatrix}
M^{-1}
\]

Hence

\[
\lim_{n \to \infty} A^n v = v \lim_{n \to \infty} A^n = v \lim_{n \to \infty} M \begin{pmatrix}
1 \\
r_1^n e^{ni\theta_1} \\
r_2^n e^{ni\theta_2} \\
\vdots \\
r_m^n e^{ni\theta_m}
\end{pmatrix}
M^{-1}
\]

-79
which converges since \( r_i < 1 \) for all \( i \).

**Problem 4.b**

We have

\[
w = M \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} M^{-1}
\]

So

\[
\lim_{n \to \infty} n^{10} |A^n v - w| = \lim_{n \to \infty} n^{10} \left( M \begin{pmatrix} 0 & \lambda_2^n \\ \vdots \\ \lambda_n^n \end{pmatrix} M^{-1} \right) = \lim_{n \to \infty} M \begin{pmatrix} 0 & n^{10} \lambda_2^n \\ \vdots \\ n^{10} \lambda_n^n \end{pmatrix} M^{-1} = 0
\]

**Problem 5**

First I claim that the eigenvalues of \( B \) must be \( \pm \sqrt{\lambda_i} \). Indeed since if \( \lambda \neq \pm \sqrt{\lambda_i} \) \( \forall i \), then

\[
Bv = \lambda v \Rightarrow B^2 v = Av = \lambda^2 v
\]

which implies that \( \lambda^2 \) is an eigenvalue of \( A \), and hence we have a contradiction. Thus the eigenvalues of \( B \) are \( \{ \pm \lambda_1, \pm \lambda_2, ..., \pm \lambda_n \} \). So \( B \) is diagonalizable since all eigenvalues are distinct.

\[
B = MD^{1/2}M^{-1}
\]

and hence

\[
A = B^2 = MDM^{-1}
\]

which implies that \( A \) and \( B \) share the same eigenvectors. Thus there are \( 2^n \) possible \( B' \)s.

**Sept 2000 Advanced Calculus**

**Problem 1**

Notice we have

\[
\sum_{n=2}^{\infty} \frac{1}{(2n-1)(2n-1)} = \sum_{n=2}^{\infty} \frac{(1/2)}{2n-1} - \frac{(1/2)}{2n+1}
\]

\[
= \frac{1}{2} \left( \frac{1}{3} - \frac{1}{5} + \frac{1}{5} - \frac{1}{7} + \frac{1}{7} + \cdots \right) = \frac{1}{6}
\]

-80
Problem 2.a
We want to show
\[ R_n = \frac{1}{n!} \int_a^x (x-t)^n f^{(n+1)}(t)dt \]
let
\[ g(t) = f(t) + f'(t)(x-t) + \frac{f''(t)(x-t)^2}{2!} + \cdots + \frac{f^{(n)}(t)(x-t)^n}{n!} \]
Then \( g(x) = f(x) \) and
\[ g(a) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)(x-a)^n}{n!} \]
Thus
\[ R_n = g(x) - g(a) = \int_a^x g' \]
notice
\[ g'(t) = f'(t) - f'(t) + f''(t)(x-t) - f''(t)(x-t) + \cdots + \frac{f^{(n-1)}(t)(x-t)^n}{n!} = \frac{f^{(n-1)}(t)(x-t)^n}{n!} \]
Thus
\[ R_n = \int_0^x \frac{f^{(n-1)}(t)(x-t)^n}{n!} dt \]

Problem 2.b
It suffices to show that
\[ \lim_{n \to \infty} e^{e^{x+1}} \frac{e^{x^{n+1}}}{(n+1)!} = 0 \]
Indeed by Stirling’s approximation, we have
\[ n! = n^n e^{-n} \sqrt{n} e^c_n \]
where \( 1 \leq c_n \leq 1 - (1/2) \log 2 \). So
\[ \lim_{n \to \infty} e^{e^{x+1}} \frac{e^{x^{n+1}}}{(n+1)!} = \lim_{n \to \infty} e^{e^{x+1}} \frac{e^{x^{n+1}} e^{n+1}}{(n+1)^{n+1} \sqrt{n+1} e^{c_n}} = \lim_{n \to \infty} e^{e^{x+1}} \frac{e^{k^{n+1}}}{(n+1)^{n+1} \sqrt{n+1}} \to 0 \]
Problem 3.a

Notice we have

\[ f'_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} 0 = 0 \]

and

\[ f'_y(0,0) = \lim_{h \to 0} \frac{(f(0,h) - f(0,0)}{h} = \lim_{h \to 0} 0 = 0 \]

Hence \( f'_x(0,0) \) and \( f'_y(0,0) \) exists.

Problem 3.b

Notice as we approach \((0,0)\) along the \( y = x \) line, we have

\[ \lim_{x \to 0} f(x,y) = \lim_{x \to 0} \frac{x^2}{2x^2} = \lim_{x \to 0} \frac{1}{2} = \frac{1}{2} \neq 0 \]

Hence it is not continuous at \((0,0)\).

Problem 3.c

Notice

\[ f(x,y) = f(r,\theta) = \begin{cases} (1/2) \sin(2\theta) & r \neq 0 \\ 0 & r = 0 \end{cases} \]

thus the set of limit points at \((0,0)\) is \( (1/2) \sin(2\theta) \in [-1/2,1/2] \).

Problem 5.a

Recall Greens Theorem: Given that \( P \) and \( Q \) are continuous in a Jordan region \( \Omega \) with boundary \( C \), we have

\[ \int \int_{\Omega} \frac{dQ}{dx}(x,y)dx + \frac{dP}{dy}(x,y)dy = \oint_{C} P(x,y)dx + Q(x,y)dy \]

I claim that

\[ \int \int_{\Omega} \frac{dP}{dy}dydx = -\oint_{C} P(x,y)dx \]

indeed since by the Fundamental Theorem of Calculus we have

\[ \int_{\phi_{2}(x)}^{\phi_{1}(x)} \left( \int_{\phi_{1}(x)}^{\phi_{2}(x)} \frac{dQ}{dx}(x,y)dy \right)dx = \int_{\phi_{2}(x)}^{\phi_{1}(x)} P(x,\phi_{2}(x)) - P(x,\phi_{1}(x))dx \]

Also lets parameterize \( C \) by \( C_{1} : (x, \phi_{1}(x)) \) and \( C_{2} : (x, \phi_{2}(x)) \) for \( a \leq x \leq b \). Thus
\[ \oint_C P(x,y)dx = \int_a^b P(x,\phi_1(x))dx + \int_b^a P(x,\phi_2(x))dx \]

\[ = \int_a^b P(x,\phi_1(x)) - P(x,\phi_2(x))dx = -\int \int_\Omega \frac{dP}{dy}dxdy \]

Likewise I claim that

\[ \int \int_\Omega \frac{dQ}{dx}dxdy = \oint_C Q(x,y)dy \]

then by the Fundamental Theorem of Calculus, we have

\[ \int \int_\Omega \frac{dQ}{dx}dxdy = \int_c^d \left( \int_{\psi_1(y)}^{\psi_2(y)} \frac{dQ}{dx} dx \right) dy = \int_c^d Q(\psi_2(y), y) - Q(\psi_1(y), y)dy \]

also notice we have

\[ \oint_C Q(x,y)dy = \int_c^d Q(\psi_1(y), y) + \int_c^d Q(\psi_2(y), y)dy = \int_c^d Q(\psi_2(y), y) - Q(\psi_1(y), y)dy = \int \int_\Omega \frac{dQ}{dx}dxdy \]

This completes the proof of Green’s Theorem.

**Problem 5.b**

By Green’s theorem, we have

\[ \oint_C (2xy - x^2)dx + (x + y^2)dy = \int_0^1 \sqrt{x} - 2x - x^2 + 2x^3 dx = \frac{1}{30} \]

Now by direct calculation, we have \( C = C_1 \cup C_2 \), where \( C_1 : (t,t^2) \) for \( t : 0 \to 1 \), and \( C_2 : (t^2,t) \) for \( t : 1 \to 0 \). Thus we have

\[ \oint_C (2xy - x^2)dx + (x + y^2)dy = \int_{C_1} (2xy - x^2)dx + (x + y^2)dy + \int_{C_2} (2xy - x^2)dx + (x + y^2)dy \]

\[ = \int_0^1 (2t^3 - t^2) + t + t^4 dt + \int_1^0 (2t^3 - t^4)2t + 2t^2 dt = \frac{7}{6} + \frac{-17}{15} = \frac{1}{30} \]

Thus we have verified Green’s Theorem.
Sept 2000 Complex Variables

Problem 1.a

Since $f$ is entire, it has a Taylor series about $z = 0,$

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$$

So we have $f(0) = a_0$ and let $h(z) = 1/(f(z) - a_0).$ Then

$$h(z) = \frac{1}{a_1 z + a_2 z^2 + \cdots} = \frac{1}{z(a_1 + a_2 z + a_3 z^2 + \cdots)}$$

notice that

$$\frac{1}{a_1 + a_2 z + a_3 z^2 + \cdots}$$

is analytic about $z = 0.$ Thus it has a Taylor series about $z = 0$

$$\frac{1}{a_1 + a_2 z + a_3 z^2 + \cdots} = b_0 + b_1 z + b_2 z^2 + \cdots$$

So

$$h(z) = \frac{b_0}{z} + b_1 + b_2 z + b_3 z^2 + \cdots$$

Now since $f$ is entire and one-to-one, by the open mapping theorem, for $|z| < 1$ exists $\delta > 0$ such that

$$\{ w : |w - a_0| < \delta \} \subset \{ f(z) : |z| < 1 \}$$

Since $f$ is one to one, $\forall z$ such that $|z| > 1,$ we have

$$|f(z) - a_0| \geq \delta$$

which implies

$$\lim_{z \to \infty} \frac{1}{|f(z) - a_0|} = |h(z)| \leq \frac{1}{\delta} < \infty$$

Hence $h$ has a removable singularity at $\infty.$ This implies that $b_1 = b_2 = b_3 = \cdots = 0.$ Since $\lim f(z) = \infty,$ $b_1 = 0.$ Hence

$$\frac{1}{f(z) - a_0} = \frac{b_0}{z}$$

which implies that

$$f(z) = a_1 z + a_0$$

and $a_1 \neq 0$ since $f$ is one to one.
Problem 1.b

Let \( f \) be such a function. Then let \( T \) be a linear fractional transformation such that \( \infty \mapsto \infty \). Then

\[
T(f(z)) = \frac{af(z) + b}{cf(z) + d}
\]

where \( T(f(z)) \) is one to one and onto of \( \mathbb{C} \rightarrow \mathbb{C} \). Hence by part (a), we have

\[
T(f(z)) = \frac{af(z) + b}{cf(z) + d} = pz + q
\]

Thus

\[
f(z) = \frac{(pd)z + (qd - b)}{(-pc)z + (a - qc)} = \frac{a'z + b'}{c'z + d'}
\]

Problem 2.a

(Ahlfors) Recall that all lines in the complex plane can be represented as \( a + bt \) where \( a, b \in \mathbb{C} \) and \( t \in \mathbb{R} \). Also the left half of the line is all \( z \) such that

\[
\Im\left(\frac{z - a}{b}\right) < 0
\]

and the right half of the line is all \( z \) such that

\[
\Im\left(\frac{z - a}{b}\right) > 0
\]

Now if \( a + bt \) is a horizontal line, then it’s trivial. Now let

\[
P(z) = A(z - \alpha_1)(z - \alpha_2)(z - \alpha_3) \cdots (z - \alpha_n)
\]

Then we have

\[
\frac{P'(z)}{P(z)} = \frac{1}{z - \alpha_1} + \frac{1}{z - \alpha_2} + \cdots + \frac{1}{z - \alpha_n}
\]

So suppose the that half plane \( H \) is defined as the part of the plane where \( \Im((z - a)/b) < 0 \). Now we know that for all \( k, \alpha_k \) is in \( H \). Suppose that \( z \notin H \), where \( P'(z) = 0 \). Then we have

\[
\Im\left(\frac{z - \alpha_k}{b}\right) = \Im\left(\frac{z - a}{b}\right) - \Im\left(\frac{\alpha_k - a}{b}\right) > 0
\]

Also recall that imaginary parts of reciprocal numbers have opposite sign. Therefore, under the same assumption, \( \Im(b/(z - \alpha_k)) < 0 \). Since this is true for all \( k \), we have

\[
\Im\left(\frac{bP'(z)}{P(z)}\right) = \sum_{k=1}^{n} \Im\left(\frac{b}{z - \alpha_k}\right) < 0
\]

hence \( P'(z) \neq 0 \), and we have a contradiction. This concludes the proof.
Problem 3.a

Clearly $f$ is one to one since if

$$f(z_1) = f(z_2) \Rightarrow z_1 + \frac{1}{z_1} = z_2 + \frac{1}{z_2} \Rightarrow z_1 - z_2 = \frac{z_1 - z_2}{z_1 \cdot z_2}$$

Now if $z_1 \neq z_2$, then we have

$$z_1 \cdot z_2 = 1$$

However this is impossible since both $|z_1|$ and $|z_2|$ is greater than 1. Hence $f$ is one-to-one. Now $f$ is analytic in $E$ and for all $z \in E$, we have

$$f'(z) = \frac{1}{2} - \frac{1}{2z^2} \neq 0$$

Hence $f$ is also a conformal mapping.

Problem 3.b

Now I claim that $f$ maps $E$ to the entire complex plane except the real interval $[-1, 1]$. For $z = x + iy \in E$

$$f(z) = \frac{1}{2} \left( (x + iy) + \frac{1}{x + iy} \right) = \frac{1}{2} \left( \frac{x(r^2 + 1)}{r^2} + iy \frac{r^2 - 1}{r^2} \right)$$

if $f(z) \in [-1, 1]$, then either $r^2 = 1$, which is not possible in $E$, or $y = 0$. But then $(1/2)(x + 1/x) \in [-1, 1]$. But that implies $x = 1$ which is not possible. Now to show that $f$ is onto, notice the preimage

$$w = \frac{1}{2}(z + 1/z) \Rightarrow z^2 - 2wz + 1 = 0$$

Hence

$$z = \frac{2w \pm \sqrt{4w^2 - 4}}{2} = w \pm \sqrt{w^2 - 1}$$

Hence if $w + \sqrt{w^2 - 1} = re^{i\theta}$, then $w - \sqrt{w^2 - 1} = (1/r)e^{-i\theta}$. Hence one of these is outside the unit circle since $r \neq 1$ (otherwise that would imply $z = \pm 1$).

Problem 3.c

The residue at $\infty$ is the residue of $f(1/z)$ at 0. Hence the residue at $\infty$ for $f$ is $1/2$. 

-86
Problem 5.a

Let \( f(z) = \log z / (1 + z^2) \), and let \( \gamma \) be the contour of two semi-circles with radius \( R \) and \( \epsilon \) in the upper-half plane. Then by the Residue Theorem, we have

\[
\int_{\gamma} f(z) \, dz = 2\pi i \frac{\log i}{2i} = i \frac{\pi^2}{2}
\]

Now notice on \( C_R \) we have \( z = Re^{i\theta} \). Hence

\[
\left| \int_{C_R} \log z \, dz \right| \leq \pi R \frac{\log(R + \pi^2)}{R^2 - 1} \leq \pi R \frac{\sqrt{\log^2 R + \pi^2}}{R^2 - 1} \to 0
\]
as \( R \to \infty \).

Now notice that on \( C_\epsilon \) we have \( z = \epsilon e^{i\theta} \). Hence

\[
\left| \int_{C_\epsilon} \log z \, dz \right| \leq \frac{\pi \epsilon \sqrt{\log^2 \epsilon + \pi^2}}{1 - \epsilon^2} \to 0
\]
as \( \epsilon \to 0 \). So as \( \epsilon \to 0 \) and \( R \to \infty \), we have

\[
\int_{-\infty}^{\infty} \log z \, dz + \int_{0}^{\infty} \frac{\log z}{1 + z^2} \, dz = i \frac{\pi^2}{2}
\]

which implies

\[
\int_{0}^{\infty} \frac{\log(-z)}{1 + z^2} \, dz + \int_{0}^{\infty} \frac{\log(z)}{1 + z^2} \, dz = i \frac{\pi^2}{2}
\]

which implies

\[
2 \int_{0}^{\infty} \frac{\log z}{1 + z^2} \, dz + i \int_{0}^{\infty} \frac{\pi}{1 + z^2} \, dz = i \frac{\pi^2}{2}
\]

Thus

\[
\int_{0}^{\infty} \frac{\log z}{1 + z^2} \, dz = 0
\]

Problem 5.b

We have two cases. **CASE 1:** Assume that \( w \) is a purely imaginary number, \( w = ib \) where \( b \in \mathbb{R} \). Then we have

\[
\int_{-\infty}^{\infty} e^{-2\pi iw} e^{-t^2/2} \, dt = \int_{-\infty}^{\infty} e^{(t-2\pi b)^2/2 + 4\pi^2 b^2/2} = e^{4\pi^2 b^2/2} \int_{-\infty}^{\infty} e^{-(t-2\pi b)^2/2} \, dt
\]

by applying \( u \)-substitution, we can see that

\[
\int_{-\infty}^{\infty} e^{-2\pi iw} e^{-t^2/2} \, dt = e^{4\pi^2 b^2/2} \sqrt{2\pi}
\]

-87
CASE 2: Now let’s assume \( w = a + bi \) where \( b \neq 0 \). Let \( f(z) = e^{-z^2/2} \) and \( \gamma \) be the contour from \(-R\) to \( R\) on the real axis, then from \( R\) to \( R + i2\pi w\), and to \(-R + i2\pi w\), and final back to \(-R\). By the Residue theorem we have

\[
\int_{\gamma} f(z) \, dz = 0
\]

So we have

\[
\int_{-R}^{R} e^{-z^2/2} \, dz + \int_{0}^{2\pi} e^{-z+iwz^2/2} \, iwdz + \int_{R}^{-R} e^{(-z+i2\pi w)^2/2} \, dz + \int_{2\pi}^{0} e^{(-R+iwz)^2/2} \, iwdz = 0
\]

Notice that

\[
\left| \int_{0}^{2\pi} e^{-(R+iwz)^2/2} \, dz \right| \leq e^{-R^2/2} \int_{0}^{2\pi} e^{Rbz + a^2z^2/2 - b^2z^2/2} \, dz \leq e^{-R^2/2} 2\pi e^{Rb^2 + a^2z^2/2} \to 0 \quad \text{as } R \to \infty.
\]

Likewise

\[
\int_{0}^{2\pi} e^{(-R+iwz)^2/2} \, iwdz \to 0 \quad \text{as } R \to \infty.
\]

So as \( R \to \infty \) we have

\[
\int_{-R}^{R} e^{-z^2/2} \, dz + \int_{-R}^{R} e^{(-z+i2\pi w)^2/2} \, dz = \int_{-R}^{R} e^{-z^2/2} \, dz + e^{2\pi^2 w^2} \int_{-\infty}^{-\infty} e^{-z^2/2 - 2\pi izw} \, dz = 0
\]

and hence

\[
\int_{-\infty}^{-\infty} e^{-z^2/2 - 2\pi izw} \, dz = \sqrt{2\pi} e^{-2\pi^2 w^2}
\]

Now let notice it does converge absolutely for all \( w \in \mathbb{C} \).

\[
\int_{-\infty}^{\infty} e^{-2\pi i tw - t^2/2} \, dt = \int_{-\infty}^{\infty} e^{-2\pi t b e^{-t^2/2}} \, dt = \int_{-\infty}^{\infty} e^{-(t + wnb)^2/2 + 4\pi^2 b^2} \, dt = e^{4\pi^2 b^2 \sqrt{2\pi}}
\]

Sept 2000 Linear Algebra

Problem 1.a

We have

\[
D = \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 \\
0 & 0 & 2 & 0 & \cdots & 0 \\
0 & 0 & 0 & 3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & 0 \\
0 & 0 & 0 & 0 & \cdots & k \\
0 & 0 & 0 & 0 & \cdots & 0
\end{pmatrix}
\]

-88
Problem 1.b

We will let $v_i = x^i$ be the bases for $i \leftarrow 0$ to $k$. So notice what $L$ does on the bases. For $j \leftarrow 0$ to $k$, we have

$$Lx^j = \sum_{p=0}^{n} \frac{1}{p!} D^p x^j$$

Now we know that $D^p x^j = j(j-1)(j-2) \cdots (j-p+1)x^{j-p}$ if $p \leq j$ and $D^p x^j = 0$ else. Thus

$$Lx^j = \sum_{p=0}^{j} \frac{j!}{p!(j-p)!} x^{j-p} = \sum_{p=0}^{j} \binom{j}{p} x^{j-p} = (1+x)^j$$

Hence dim(ker($L - T$)) = $k + 1$.

Problem 2.a

Notice for

$$V_1 = \{ Bx : x \in \mathbb{R}^n \}$$

we have dim($V_1$) + $n(V_1) = n$ where $n \geq 0$. Hence

$$\text{dim}(V_1) = \text{Rank}(B) \leq n$$

Now let $V_2 = \{ Ax : x \in V_1 \}$. Then dim($V_2$) + $n(V_2) = \text{dim}(V_1) = \text{Rank}(B)$. Also $\text{Rank}(AB) = \text{dim}(V_2)$. SO we have

$$\text{Rank}(A) = \text{Rank}(AB) \leq \text{Rank}(B)$$

Problem 2.b

It suffices to show that $\text{Rank}(AB) \leq \text{Rank}(A)$. Suppose $\text{Rank}(AB) > A$. Then $\exists x \in \mathbb{R}^n$ such that for

$$A = \begin{pmatrix} \left| \right| & \cdots & \left| \right| \\ c_1 & \cdots & c_p \\ \left| \right| & \left| \right| & \left| \right| \end{pmatrix}$$

Then $ABx \notin S(c_1, \ldots, c_p)$. But

$$ABx = \begin{pmatrix} \left| \right| & \left| \right| \end{pmatrix} Bx = \begin{pmatrix} a_1 \\ \vdots \\ a_p \end{pmatrix} \in S(a_1, \ldots, a_p)$$

Hence we have a contradiction and thus $\text{Rank}(AB) \leq \text{Rank}(A)$.
Problem 3.a

If $A$ satisfies $(A - I)(A - 2I)^2 = 0$, then

$$(A - I)(A - 2I)^2 x = 0 \quad \forall x \in \mathbb{R}^n$$

If $Ax = \lambda x$, then the above is $(\lambda - 1)(\lambda - 2)^2 x$. So the only possible eigenvalues are 1 and 2. So $A - \lambda I$ just bumps down the generalized eigenvectors. So $A$ cannot have a Jordan Block of size $> 1$ for $\lambda = 1$ and cannot have a Jordan Block of size $> 2$ for $\lambda = 2$. So the 9 possible Jordan Blocks are

$$
\begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix},
\begin{pmatrix}
2 & 2 \\
2 & 2
\end{pmatrix},
\begin{pmatrix}
1 & 2 \\
2 & 2
\end{pmatrix},
\begin{pmatrix}
1 & 1 \\
2 & 2
\end{pmatrix},
\begin{pmatrix}
2 & 1 \\
2 & 2
\end{pmatrix},
\begin{pmatrix}
1 & 2 \\
2 & 2
\end{pmatrix},
\begin{pmatrix}
1 & 1 \\
2 & 2
\end{pmatrix},
\begin{pmatrix}
1 & 2 \\
2 & 2
\end{pmatrix},
\begin{pmatrix}
2 & 2 \\
2 & 2
\end{pmatrix}
$$

Remember that 2 matrices are similar if they have the same Jordan Block. So describing the number of conjugacy class is equivalent to describing the number of Jordan blocks.

Problem 4

We have

$$\det(A - \lambda I) = \begin{pmatrix} 1 - \lambda & 2 \\ 4 & 1 - \lambda \end{pmatrix} = \lambda^2 - 2\lambda - 7 = 0$$

Then we have $\lambda_1 = 1 + 2\sqrt{2}$ and $\lambda_2 = 1 - 2\sqrt{2}$. Then we find our eigenvectors $v_1 = (1, \sqrt{2})^T$ and $v_2 = (1, -\sqrt{2})^T$. Now notice that $S(v_1, v_2) = \mathbb{R}^2$. Hence

$$A^n v = c_1 \frac{A^n}{2^n} \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} + c_2 \frac{A^n}{2^n} \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} = c_1 \left( \frac{1 + 2\sqrt{2}}{2} \right)^n \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} + c_2 \left( \frac{1 - 2\sqrt{2}}{2} \right)^n \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$$

So when $c_1 = 0$, then series converges. Therefore $v = c_2(1, -\sqrt{2})^T$ for it to converge.

Problem 5.a

Let $A = (v_1, v_2, ..., v_n)$. Then $M = A^T A$ and

$$x^T M x = x^T A^T A x = (Ax)^T A x = |Ax|^2 \geq 0$$

Problem 5.b

We know that $A^T A$ is invertible if and only if the columns of $A$ are linearly independent.
Problem 5.c

Notice when we perform Gram Schmidt of $v_i$, we have

$$e_n = v_1 - \frac{(v_1, v_n)}{(v_1, v_1)}v_1 - \frac{(v_2, v_n)}{(v_2, v_2)}v_2 - \cdots - \frac{(v_{n-1}, v_n)}{(v_{n-1}, v_{n-1})}v_{n-1}$$

So

$$\begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix} = \begin{pmatrix} v_1 & \cdots & v_{n-1} & e_n \end{pmatrix} \begin{pmatrix} 1 & \mathbf{X} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{X} & \cdots & \mathbf{X} & 1 \end{pmatrix}$$

So

$$A^T A = \begin{pmatrix} 1 & \vdots \\ \vdots & \ddots & \vdots \\ \mathbf{X} & \cdots & \mathbf{X} & 1 \end{pmatrix} \begin{pmatrix} - & v_1 & - \\ - & \vdots & - \\ - & e_n & - \end{pmatrix} \begin{pmatrix} v_1 & \cdots & v_{n-1} & e_n \end{pmatrix} \begin{pmatrix} 1 & \mathbf{X} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{X} & \cdots & \mathbf{X} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \vdots \\ \vdots & \ddots & \vdots & \ddots \\ \mathbf{X} & \cdots & \mathbf{X} & 1 \end{pmatrix} \begin{pmatrix} M_{n-1} & 0 & 0 \\ 0 & 0 & e_n^T e_n \end{pmatrix} \begin{pmatrix} 1 & \mathbf{X} \\ \vdots & \ddots & \vdots & \ddots \\ \mathbf{X} & \cdots & \mathbf{X} & 1 \end{pmatrix}$$

So $\det(M_n) = |e_n|^2 \det(M_{n-1})$. 