SECTION 11.4

1. diverges; \( a_k \neq 0 \)

2. (a) \( \sum |a_k| = \sum \frac{1}{2k} \) diverges, so not absolutely convergent.
   (b) \( \frac{1}{2(k+1)^2} < \frac{1}{2k^2} \); \( a_k \to 0 \); converges conditionally; Theorem 11.4.3.

3. diverges; \( \frac{1}{k + 1} \to 1 \neq 0 \)

4. (a) \( \sum |a_k| = \sum \frac{1}{k!} \); does not converge absolutely.
   (b) converges conditionally; Theorem 11.4.3.

5. (a) does not converge absolutely; integral test,
   \( \int_1^\infty \frac{\ln(x)}{x^2} \, dx = \lim_{k \to \infty} \left[ \frac{\ln(x)}{x^2} \right]_1^k = \infty \)
   (b) converges conditionally; Theorem 11.4.3.

6. diverges; \( a_k \neq 0 \)

7. diverges; limit comparison with \( \sum \frac{1}{k} \); another approach:
   \( \sum \frac{1}{k} - \frac{1}{k^2} \) diverges since \( \sum \frac{1}{k} \) diverges and
   \( \sum \frac{1}{k^2} \) converges.

8. converges absolutely (terms already positive); ratio test,
   \( \frac{a_{k+1}}{a_k} = \frac{(k+1)^{k+1}}{k^k} = \left( \frac{k+1}{k} \right)^{k+1} \left( \frac{k+1}{k} \right)^{1/2} = \frac{1}{2} \left( \frac{k+1}{k} \right)^{1/2} \)

9. (a) does not converge absolutely; limit comparison with \( \sum \frac{1}{k} \);
   (b) converges conditionally; Theorem 11.4.3.

10. converges absolutely by ratio test.

11. diverges; \( a_k \neq 0 \)

12. diverges; \( a_k \neq 0 \)

13. (a) does not converge absolutely;
   \( (\sqrt{k+1} - \sqrt{k}) \left( \frac{\sqrt{k+1} + 1}{\sqrt{k+1} + \sqrt{k}} \right) = \frac{1}{\sqrt{k+1} + \sqrt{k}} \)
   and
   \( \sum \frac{1}{\sqrt{k+1} + \sqrt{k}} > \sum \frac{1}{2\sqrt{k+1}} = \frac{1}{2} \sum \frac{1}{\sqrt{k+1}} \) (a p-series with \( p < 1 \))
   (b) converges conditionally; Theorem 11.4.3.
14. (a) does not converge absolutely: \( \frac{k}{k^2 + 1} \geq \frac{k}{2k^2} = \frac{1}{2k} \) \hspace{1cm} \text{comparison with} \hspace{1cm} \sum \frac{1}{2k}

(b) \( \frac{k+1}{(k+1)^2 + 1} < \frac{k}{k^2 + 1} \) \hspace{1cm} \text{converges conditionally; Theorem 11.4.3.}

15. converges absolutely (terms already positive); basic comparison,

\[ \sum \sin \left( \frac{x}{4k^2} \right) \leq \sum \frac{x}{4k^2} = \frac{x}{4} \sum \frac{1}{k^2} \quad (|\sin x| \leq |x|) \]

16. (a) does not converge absolutely:

\[ \sum \frac{1}{\sqrt{k(k+1)}} > \sum \frac{1}{k+1} \]

(b) converges conditionally by Theorem 11.4.3

17. converges absolutely; ratio test, \( \frac{a_{n+1}}{a_k} = \frac{k+1}{2k} \quad \frac{1}{2} \quad \text{with} \quad \sum \frac{1}{k^{\sqrt{2}}} \)

18. terms all positive, converges absolutely: \( a_k = \frac{1}{\sqrt{k(k+1)}} \), comparison with \( \sum \frac{1}{k^{\sqrt{2}}} \)

19. (a) does not converge absolutely; limit comparison with \( \sum \frac{1}{k} \)

(b) converges conditionally; Theorem 11.4.3

20. (a) does not converge absolutely:

\[ \frac{k+2}{k^2 + k} > \frac{k}{2k^2} = \frac{1}{2k} \]

(b) converges conditionally; Theorem 11.4.3

21. diverges; \( a_k = \frac{x^{4-2}}{e^k} = \frac{1}{16} \left( \frac{e}{x} \right)^k \neq 0 \)

22. converges absolutely by integral test:

\[ \int_{1}^{\infty} x^{2-1} \, dx \quad \text{converges} \]

23. diverges; \( a_k = k \sin(1/k) = \frac{\sin(1/k)}{1/k} \rightarrow 1 \neq 0 \)

24. converges absolutely; ratio test,\( \frac{|a_{n+1}|}{a_k} = \frac{(k+1)(k+1)!}{k^2} \frac{k!}{(k+1)!} \frac{1}{k} \rightarrow 0 \), so \( a_k \neq 0 \)

25. converges absolutely; ratio test, \( \frac{|a_{n+1}|}{a_k} = \frac{(k+1)^{k+1} \cdot e^{-(k+1)}}{k^k} \rightarrow \frac{k+1}{e} \)

26. (a) \[ \sum \frac{(1)^k}{k} \]

(b) converges conditionally; Theorem 11.4.3

27. diverges; \( \sum (1)^k \cos \frac{1}{k} \frac{1}{k} \neq \sum \frac{1}{k} \)

28. Converges absolutely; \( |a_k| = \left| \frac{\sin(\pi k/2)}{\pi k} \right| < \frac{1}{k^{\sqrt{2}}} \)

29. converges absolutely; basic comparison
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\[ \sum_{k=1}^{\infty} \frac{\sin(\pi k/4)}{k^2} \leq \sum_{k=1}^{\infty} \frac{1}{k^2} \]

30. The series \( \sum \left( \frac{1}{3k+2} - \frac{1}{3k+3} \right) \) converges by comparison with \( \sum \frac{1}{k^2} \).

If \( \sum \left( \frac{1}{3k+2} - \frac{1}{3k+3} \right) \) converges, then

\[ \frac{1}{3k+1} \sum \left( \frac{1}{3k+2} - \frac{1}{3k+3} \right) - \sum \left( \frac{1}{3k+2} - \frac{1}{3k+3} \right) \]

would converge, which is not the case.

31. diverges: \( a_k \neq 0 \)

32. error \( < a_{21} = \frac{1}{21} \)

33. Use (11.4.1): \( | s - s_n | < a_n \).

34. error \( < a_5 = \frac{1}{10^5} = 0.00001 \)

35. Use (11.4.1): \( | s - s_n | < a_n \).

36. error \( < a_{n-1} \).

37. \( \frac{10}{11} \) geometric series with \( a = 1 \) and \( r = \frac{1}{10} \):

\[ \sum_{n=1}^{\infty} a_n = 1 - \frac{1}{10} \]

38. \( (0.9)^{N+1} \cdot \frac{1}{N+1} < 0.001 \)

39. Use (11.4.1): \( | s - s_n | < a_n \).

40. (a) \( \sum_{k=1}^{\infty} \frac{(-1)^k}{(k+1)!} \approx 0.841 \) \( \approx 0.841471 \)

(b) \( \sum_{n=1}^{\infty} \frac{(-1)^{k+1}}{k} \approx 0.693 \) \( \approx 0.693147 \)

(c) \( \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} \approx 0.657 \) \( \approx 0.657959 \)

41. \( n = 999; \) \( L = \sum_{k=1}^{999} \frac{k^2}{k^2 + 1} \)

42. The series diverges because among the partial sums are all sums of the form

\[ \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \]

Then, for instance,

\[ s_1 = \frac{1}{2}, \quad s_2 = \frac{1}{2} + \frac{1}{3} = s_1 + \frac{1}{3}, \quad s_3 = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = s_2 + \frac{1}{4}, \quad \text{and so on.} \]

This does not violate the theorem on alternating series because, in the notation of the theorem, it is not true that \( \{a_n\} \) decreases.