Practice problems, Calculus II, Spring 2006.

**Complex numbers**

1. Roots and powers.
   
   (a) Find the two square roots of $z = 1 + i$.
   
   (b) Find the three cube roots of $z = 4(-1 + \sqrt{3}i)$.
   
   (c) Let $z = 1 - \sqrt{3}i$. Compute $z^6$.

2. Describe geometrically the regions of the complex plane determined by:
   
   (a) $|5z - 2| < 1$.
   
   (b) $|2z + 3i| > 1$.
   
   (c) $\text{Re} [(1 + i)z] < 0$.

**Chapter 9**

1. Find the arc length of the curve $f(x) = \frac{x^3}{6} + \frac{1}{2x}$, $x \in [1, 3]$.

2. Express the curve by an equation in $x$ and $y$: $x(t) = 3 + \cos t$, $y(t) = 3 - 2 \sin t$.

3. Find the area of the region that is common to $r = a \cos 3\theta$ and $r = \frac{a}{2}$. Take $a > 0$.

4. Identify the curve given by $r = \frac{10}{2 + \cos \theta}$ and write the equation in rectangular coordinates.

5. Write the equation $x + y^2 = x - y$ in polar coordinates.

6. Find the equation in $x$ and $y$ for the line tangent to the curve $x(t) = 3t$, $y(t) = t^2 - 1$ at $t = 1$.

7. The equations $x(t) = t^2 + 2$, $y(t) = t^3 - 3$ gives the position of a particle at time $t$ from $t = 0$ to $t = 1$. Find the initial speed of the particle, the terminal speed, and the distance travelled.

8. Calculate $\frac{dy}{dx}$ at the point $t = 1$ without eliminating the parameter if $x(t) = e^t - 1$ and $y(t) = 3 + e^{2t}$.

9. Sketch and identify the polar curve $r = 1 + \cos \theta$.

10. Find the area of the region enclosed by $r = 2 \cos 3\theta$. 
Chapter 10

1. Find the limit if the sequence converges:
   (a) $\frac{2^n}{4^n+1}$
   (b) $\frac{4n}{\sqrt{n^2+1}}$
   (c) $\frac{(2n+1)^2}{(3n-1)^2}$
   (d) $(1 + \frac{1}{n})^{n/2}$
   (e) $2\ln(3n) - \ln(n^2 + 1)$

2. Find the limits as $n \to \infty$:
   (a) $(1 + \frac{x}{n})^{3n}$
   (b) $(1 + \frac{1}{n})^n$
   (c) $(1 + \frac{1}{n})^{n^2}$

3. (a) $\lim_{x \to 0} \frac{1+x-e^x}{x(e^x-1)}$
   (b) $\lim_{x \to 0} \left[ \frac{1}{\sin x} - \frac{1}{x} \right]$
   (c) $\lim_{n \to \infty} \left( \frac{1}{n} \ln \frac{1}{n} \right)$

4. Evaluate:
   (a) $\int_{e}^{\infty} \frac{\ln x}{x} \, dx$
   (b) $\int_{-3}^{3} \frac{dx}{x(x+1)}$

5. Determine whether (and why) the following diverge or converge:
   (a) $\int_{1}^{\infty} \frac{x}{\sqrt{1+x^3}} \, dx$
   (b) $\int_{0}^{\infty} \frac{dx}{(1+x^3)^{1/6}}$
   (c) $\int_{1}^{\infty} \frac{\ln x}{x^2} \, dx$

Chapter 11

1. Find the sum of the series:
   (a) $\sum_{k=0}^{\infty} \frac{(-1)^k}{2^k}$
   (b) $\sum_{k=0}^{\infty} \frac{1}{(k+1)(k+3)}$
   (c) $\sum_{k=0}^{\infty} (-1)^k x^{2k}$
(d) \[ \sum_{k=0}^{\infty} \frac{1}{k!} x^{2k+1}. \]

2. Determine if the following series converges or diverges:

(a) \[ \sum_{k=0}^{\infty} \frac{\ln(k)}{k^3} \]
(b) \[ \sum_{k=0}^{\infty} \frac{\ln(k)}{k^{3/4}} \]
(c) \[ \sum_{k=0}^{\infty} k^2 \frac{2^{-k^3}}{k^3} \]
(d) \[ \sum_{k=0}^{\infty} \frac{2^k k!}{k^3} \]
(e) \[ \sum_{k=0}^{\infty} k^2 x^k. \]

3. Check the conditional and absolute convergence of:

(a) \[ \sum_{k=0}^{\infty} (-1)^k \frac{(k!)^2}{(2k)!} \]
(b) \[ \sum_{k=0}^{\infty} (-1)^k \frac{k}{k \ln k} \]
(c) \[ \sum_{k=0}^{\infty} (-1)^k \frac{k^2}{2^k} \]

4. Find the interval of convergence of:

(a) \[ \sum_{k=0}^{\infty} \frac{2^k}{k^2} x^k. \]
(b) \[ \sum_{k=0}^{\infty} \frac{1}{\ln k} x^k. \]

5. Expand \( f(x) \) in powers of \( x \):

(a) \( f(x) = x \cos(x^2) \),
(b) \( f(x) = \frac{1-x}{1+x} \).
(c) \( f(x) = \ln(2 + 3x) \). Expand around \( x = 4 \) and specify the values of \( x \) for which the expansion is valid.
(d) \( f(x) = \frac{1}{\sqrt{1+x^2}} \). Explicitly write out the first five terms. (Hint: binomial series).

Chapter 18 (18.1-18.3).

For section 18.4, see suggested HW problems.

1. Solve \( y' + \frac{2}{x} y = \frac{y^3}{x^2} \)

2. An object with mass \( m \) is dropped from rest and we assume that the air resistance is proportional to the speed of the object. If \( s(t) \) is the distance dropped after \( t \) seconds, then the speed is \( v = s'(t) \) and the acceleration is \( a = v'(t) \). If \( g \) is the
acceleration due to gravity, then the downward force on the object is $mg - cv$, where $c$ is a positive constant, and Newton’s Second Law gives

$$m \frac{dv}{dt} = mg - cv.$$ 

Solve for $v(t)$, what is the velocity as $t \to \infty$, find the distance the object has fallen after $t$ seconds.

3. Solve the IVP: $\frac{dv}{dt} - 2tv = 3t^2 e^{t^2}$, $v(0) = 5$.

4. Solve $\frac{dy}{dx} = y^2 + 1$, $y(1) = 0$

5. The Pacific halibut fishery has been modeled by the differential equation

$$\frac{dy}{dt} = ky \left(1 - \frac{y}{K}\right)$$

where $y(t)$ is the biomass (total mass of the members of the population) in kilograms at time $t$ (measured in years), the carrying capacity is estimated to be $K = 8 \times 10^7$ kg, and $k = 0.71$ per year. If $y(0) = 2 \times 10^7$ kg, find the biomass a year later. How long will it take for the biomass to reach $4 \times 10^7$ kg?

6. Find the integral curve $(xe^y - e^x)y' = ye^x - e^y$

7. Solve $(x - 2)y' + y = 5(x - 2)^2 y^{1/2}$

8. Find the integral curve that satisfies $y' = \frac{y^3 - x^3}{xy}$, $y(1) = 2$

9. Solve $xy' + 2y = \frac{\cos x}{x}$

10. Solve $yy' = 4x \sqrt{y^2 + 1}$

11. Find the integral curves $xy' = x^2(e^y)^{1/x} / y + y$

**Solutions**

**Complex numbers**

1. (a) $2^{1/4}(\cos(\pi/8) + i \sin(\pi/8))$, $2^{1/4}(\cos(9\pi/8) + i \sin(9\pi/8))$.

   (b) $2(\cos(2\pi/9) + i \sin(2\pi/9))$, $2(\cos(8\pi/9) + i \sin(8\pi/9))$, $2(\cos(14\pi/9) + i \sin(14\pi/9))$.

   (c) 64.

2. (a) Interior of circle w radius 1/5 centered in 2/5.

   (b) Exterior of circle w radius 1/2 centered in $-3/2$ i.

   (c) The half complex plane s.t $\text{Im}(z) > \text{Re}(z)$.
Chapter 9
1. \( \frac{14}{3} \)
2. ellipse: \( (x - 3)^2 + \frac{(y-3)^2}{4} = 1; \)
3. \( a^2 \left( \frac{x}{6} - \frac{\sqrt{3}}{8} \right) \)
4. ellipse: \( \frac{(x+10/3)^2}{400/9} + \frac{y^2}{400/12} = 1; \)
5. \( r = \frac{\cos \theta - \sin \theta}{1 + 2 \sin \theta \cos \theta} \)
6. \( 2x - 3y = 6. \)
7. initial speed= 0, terminal speed= \( \sqrt{3} \), distance= \( \frac{1}{27} (13\sqrt{3} - 8) \).
8. 2.
9. \( 1 + \cos \theta. \)

Chapter 10
1. (a) 0
   (b) 4
   (c) 4/9
   (d) \( \sqrt{e} \)
   (e) \( \ln 9 \)
2. (a) \( e^{3x} \)
   (b) 1
   (c) \( \infty \)
3. (a) \(-1/2 \)
   (b) \( 1/3 \)
   (c) 0
4. (a) diverges
   (b) diverges
5. (a) converges
   (b) diverges
   (c) converges
Chapter 11

1. (a) Geometric series with \( x = -1/5 \); the sum is \( 5/6 \).
   (b) Use partial fraction decomposition. Almost all terms cancel. The sum is \( 3/4 \).
   (c) Geometric series for \(-x^2\). The sum is \( 1/(1 + x^2) \).
   (d) The series is \( x \sum_{k=0}^{\infty} \frac{1}{k!} x^{2k} = x \sum_{k=0}^{\infty} \frac{1}{k!} (x^2)^k \). This is \( x \) times the Taylor series for \( e^{x^2} \), i.e. the sum is \( xe^{x^2} \).

2. Remember: there if often more than one way to obtain the answer.
   (a) Converges. For example basic comparison test with \( 1/k^2 \).
   (b) \( \sum_{k=0}^{\infty} \frac{\ln(k)}{k^{3/4}} \) Converges. Integral test.
   (c) Converges. Integral test.
   (d) \( \sum_{k=0}^{\infty} \frac{2e^k}{k} \) Converges. Ratio test gives \( 2(k/(k+1))^k \), which has the limit \( 2e^{-1} < 1 \) (use L’Hopital).
   (e) Converges. Ratio test.

3. (a) Converges absolutely and hence also conditionally (ratio test).
   (b) Converges conditionally but not absolutely (integral test).
   (c) Converges absolutely and hence also conditionally (ratio test).

4. (a) \(-1/2 \leq x \leq 1/2 \).
   (b) \(-1 \leq x < 1 \).

5. Expand \( f(x) \) in powers of \( x \):
   (a) \( \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{4k+1} \) (Can use the known expansion for \( \cos(y) \) to obtain this).
   (b) This is \( (1-x) \) times the sum of the geometric series for \(-x \). \( \sum_{k=0}^{\infty} (-1)^k (x^k - x^{k+1}) \).
   (c) \( f(x) = \ln 14 + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \left( \frac{3}{4} \right)^k (x - 4)^k \). Valid for \(-2/3 < x \leq 26/3 \).
   (d) \( (1 + x^2)^{-1/2} = \sum_{k=0}^{\infty} \left( -\frac{1}{2} \right)^k x^{2k} \). First five terms: \( 1 - \frac{1}{2} x^2 + \frac{3}{8} x^4 - \frac{5}{16} x^6 + \frac{35}{128} x^8 \).

Chapter 18 (18.1-18.3)

1. \( y = \pm[Cx^4 + 2/(5x)]^{-1/2} \)

2. \( v(t) = \frac{mg}{c}(1 - e^{-ct/m}) \),
   \( v(t \to \infty) = mg/c \),
   \( s(t) = (mg/c)[t + (m/c)e^{-ct/m}] - m^2 g/c^2 \)
3. $v = t^3 e^t + 5e^t$

4. $y = \tan(x - 1)$

5. $3.23 \times 10^7$ kg, it will take about 1.55 years.

6. $xe^y - ye^x = C$

7. $y = \left[(x - 2)^2 + \frac{C}{\sqrt{x-2}}\right]^2$

8. $y^3 + 3x^3\ln|x| = 8x^3$

9. $y = \frac{\sin x}{x^2} + \frac{C}{x^2}$

10. $y = \pm \sqrt{(2x^2 + C)^2 - 1}$

11. $y + x = xe^{y/x}[C - \ln x]$