1. Use the first derivative sign test to find the maximum of \( y = xe^{-2x} \), where \( x \geq 0 \).

2. A piece of wire 50 centimeters long is to be cut into two pieces. One piece will be formed into a square and the other into a circle. How should the wire be cut so that the combined area of the circle and the square is maximum? Hint: let \( x \) be the length used for the circle \((0 \leq x \leq 50)\). After you find the critical points be sure and check what happens at the endpoints, i.e. at \( x = 0 \) and \( x = 50 \), which correspond to all the wire being used for the square or all for the circle, respectively.

Solutions:

1. Throughout this problem we will write \( y = f(x) = xe^{-2x} \). To find the maximum of a function using the first derivative test we need to find the critical points, which if you recall are points where \( y' = f'(x) = 0 \). By the product rule, chain rule and a little bit of factoring we have

\[
y' = e^{-2x} + xe^{-2x}(-2) = e^{-2x}(1 - 2x)
\]

Therefore if \( y' = e^{-2x}(1 - 2x) = 0 \) then we must have either \( e^{-2x} = 0 \) or \( 1 - 2x = 0 \). But \( e \) to any power is never zero, so we must have \( 1 - 2x = 0 \) which means \( x = 1/2 \). Thus \( x = 1/2 \) is a critical point, and at this point the function \( y = f(x) \) has a possible maximum or minimum, but we don’t know which. The first derivative test by itself does not provide enough information to determine if we have found a max or a min. Therefore we need a second test and, coincidentally enough, it just happens to be called the second derivative test which we do below. For this quiz it was not necessary to include this as part of your answer, but on future tests you should always write it down.

To apply this test we of course need to know the second derivative. We already know the first derivative, so we find the second by taking the derivative of the derivative. We have

\[
y'' = f''(x) = \frac{d}{dx}(e^{-2x}(1 - 2x))
\]

\[
= -2e^{-2x}(1 - 2x) + 2e^{-2x}
\]

\[
= e^{-2x}(4x - 4)
\]
Our next step is to find the sign on the second derivative at our critical point. At $x = 1/2$ we have $y'' = f''(x) = e^{-2}(2 - 4) = -2e^{-2} < 0$. ($e$ to any power is always positive) so the second derivative is negative at $x = 1/2$. This tells us, and it may seem counterintuitive, that $y = f(x)$ has a maximum at $x = 1/2$. If you’re confused as to why the second derivative is negative at a maximum remember this: the sign on the second derivative tells us the concavity of the graph, and at a maximum the graph is always concave down. But concave down means the same as a negative second derivative.

There’s one last step involved in this problem. Note the question asks for the maximum of the function. The point $x = 1/2$ is not the maximum but the point at which the maximum occurs. To find the maximum we have to plug in $x = 1/2$ to the function and get $y = f(1/2) = \frac{1}{2}e^{-2(1/2)} = \frac{1}{2}e^{-1}$ as our maximum value.

2. As in most word problems the most important step is drawing a useful picture.

Thus the wire of length 50 cm is cut into a circle of circumference $x$ cm and a square of perimeter $50 - x$ cm. We need a formula for the area enclosed by the two figures. Since the square has perimeter $50 - x$ each side must be of length $(50 - x)/4$. The area of the square is therefore $((50 - x)/4)^2$. For the circle recall the formulas relating the
circumference $C$, the radius $r$ and the area $A$: $C = 2\pi r$, $A = \pi r^2$. The first formula tells us that $r = C/(2\pi)$, so plugging this into the second formula we get

\[
A = \pi \left( \frac{C}{2\pi} \right)^2 = \frac{C^2}{4\pi^2} = \frac{C^2}{4\pi}
\]

since $C = x$. The total area of the circle and square is therefore given by the formula

\[
A = A(x) = \frac{x^2}{4\pi} + \frac{(50 - x)^2}{16}
\]

From here on the problem is easy, all we need to do is find the maximum of this function, keeping in mind that we must have $0 \leq x \leq 50$. Differentiating we get

\[
A'(x) = \frac{2x}{4\pi} + \frac{2(50 - x)(-1)}{16} = \frac{x}{2\pi} - \frac{50 - x}{8}
\]

Now to find the critical points we have to solve $A'(x) = 0$. We have

\[
A'(x) = \frac{x}{2\pi} - \frac{50 - x}{8} = \frac{4x - \pi(50 - x)}{8\pi} = 0
\]

which is possible only if $4x - \pi(50 - x) = 0$, i.e.

\[
4x - \pi(50 - x) = (4 + \pi)x - 50\pi = 0
\]

\[
x = \frac{50\pi}{4 + \pi} \approx 22
\]

So 22 is a critical point. The area when $x = 22$ is

\[
A(22) = \frac{22^2}{4\pi} + \frac{(50 - 22)^2}{16} \approx 87.52
\]

Now we're only left to wonder if this is in fact the maximum area possible. Since it's the only critical point it could very well be, but it could just as easily be the minimum area as well. If we plug in $x = 0$ to our area formula, which in the context of this problem means use all of the wire to make the square, we get

\[
A(0) = \frac{0^2}{4\pi} + \frac{50^2}{16} = 156.25
\]

which is already bigger than the area at $x = 22$. If we plug in $x = 50$, i.e. we use all the wire to make the circle, we get

\[
A(0) = \frac{50^2}{4\pi} + \frac{0^2}{16} \approx 198.94
\]
So at $x = 22$ we don’t have a maximum at all, we actually have a minimum. The maximum area enclosed by the square and circle is actually 198.94, which we get by using the entire 50 cm to make the circle and forgetting about the square. This underlies the importance of always checking the endpoint values of $x$ for possible max or min. Had we not done it here we probably would have guessed $x = 22$ as the point at which the area is maximized, which we now see is totally incorrect.