Problem Set 8

Answers to the underlined problems are written up on the following pages.
Section 4.1: 3, 5, 10, 15, 17; 1(iii), 4, 6, 12, 14, 16
Section 4.2: 18, 39, 40, 41, 45–48; 1–4, 13–16, 20, 22, 23–26, 34–37
Section 4.3: 1, 3, 4, 5, 7, 19, 21, 26; 2, 8, 9, 16, 17, 24
Section 4.4: 3, 5, 7, 9, 15; 2, 4, 8, 10

Section 4.1

3 \( t = 2, r = 0.15, P = $100 \)
   (i) Use \( A = P(1 + \frac{r}{n})^{nt} \) where \( N = 1 \). \( A = $100(1 + \frac{0.15}{1})^2 = $132.25 \)
   (ii) \( N + 4, A = 100(1 + \frac{0.15}{4})^{4t} = $134.25. \)
   (iii) Use \( A = Pe^{rt}. \) \( A = $100e^{0.15 \cdot 2} = 100e^3 = $134.99. \)
   If we start with $10, all answers are one-tenth our calculated answers for \( P = $100. \) If we start with $1000, all answers are ten times our calculated answers for \( P = $100. \)

5 What would a $9300 savings account investment return after one year?
   \( t = 1, N = 4, r = 0.07, \) so \( A = $9300(1 + \frac{0.07}{4})^{1 \cdot 4} = $9968.29. \) It’s better to buy the bond.

10 \( A = $3000, N = 1, t = 5, r = 0.08. \) $3000 = \( P(1 + \frac{0.08}{4})^5 \approx P(1.46933) \Rightarrow \) \( P = \frac{3000}{1.46933} \approx $2041.75 \)

15 \( P(N) = (1-r)^N \) where \( N = 4, P = 8500, r = 0.3. \) \( P(4) = 8500(1-0.3)^4 \approx $2040.85. \)

17 \( t = 135 \) years, \( P = 1, r = 0.05. \)
   (i) \( A = $1(1 + \frac{0.05}{135})^{135} = $725.36 \) (ii) \( N = 4: \) \( A = 1(1 + \frac{0.05}{135})^{4 \cdot 135} = $819.06 \) (iii) \( A = Pe^{rt} = 1 \cdot e^{0.05(135)} = $854.06 \)

Section 4.2

18 \( t = 4.5, r = 0.06, P = $10,000. \) \( A = Pe^{rt} = 10,000e^{0.06(4.5)} = $13,099.64. \)

39 \( A = 2P, t = 8. \) \( 2P = Pe^{r \cdot 8} \Rightarrow 2 = e^{r \cdot 8} \Rightarrow \ln 2 = \ln e^{8r} = 8r \Rightarrow r = \ln \frac{2}{8} \approx 0.0866 \) or 8.66%

40 \( A = 15000, P = 5000, t = 13 \) years, so \( $15,000 = 5,000e^{13r} \Rightarrow 3 = e^{13r} \Rightarrow r = \frac{\ln 3}{13} \approx 0.0845 = 8.45\% \)

41 \( N(t) = 8 \cdot 2^{1.6t}. \) In the beginning \( (t = 0), N = 8 \cdot 2^0 = 8 = \) original pheasant population. We want \( t \) for \( N(t) = 24 \Rightarrow 24 = 8 \cdot 2^{1.6t} \Rightarrow 3 = 2^{1.6t} \Rightarrow \ln 3 = \ln 2^{1.6t} = 1.6t \ln 2. \) Thus
   \[ t = \frac{\ln 3}{1.6 \ln 2} \approx 0.991 \approx 1 \) year.

45 \( A(t) = 2^{-t/T}. \) We want \( e^{kt} = 2^{-t/T} \Rightarrow \ln e^{kt} = \ln 2^{-t/T} \Rightarrow kt = -\frac{t}{T} \ln 2 \Rightarrow k = -\frac{\ln 2}{T} \Rightarrow A(t) = e^{-t \ln 2/T}. \)

46 \( P(t) = 27,000,000 \cdot 2^{t/30} \Rightarrow 54,000,000 = 27,000,000 \cdot 2^{t/30}. \) We can see by inspection that \( t = 30 \) satisfies this. To be sure, we check: \( 2 = 2^{t/30} \Rightarrow \ln 2 = \ln 2^{t/30} = \frac{t}{30} \ln 2 \Rightarrow t = 30. \)

47 \( P(t) = 3P(0) = P(0) \cdot 2^{t/30} \Rightarrow \ln 3 = \ln 2^{t/30} = \frac{t}{30} \ln 2 \Rightarrow t = 30 \frac{\ln 3}{\ln 2} \approx 47.55. \)

48 \( 100,000,000 = 27,000,000 \cdot 2^{t/30} \Rightarrow \ln \left( \frac{1000}{27} \right) = \ln 2^{t/30} = \frac{t}{30} \ln 2 \Rightarrow t = \frac{30(\ln 100 - \ln 27)}{\ln 2} = \frac{30 \ln(\frac{100}{27})}{\ln 2} \approx 56.67 \) years.
Section 4.3

1. $A = $100, \( t = 10 \), \( r = 0.06 \), \( N = 2 \) for second part of problem

\[ P = Ae^{-rt} = \$100e^{-0.06(10)} = \$54.88; \quad P = A(1 + \frac{r}{N})^{-Nt} = \$100(1 + \frac{0.06}{2})^{-2(10)} \approx \$55.37 \]

3. See what return an \$8400 investment in a savings account would bring after 2.5 years.

\[ P = \$8400, \ r = 0.0818, \ t = 2.5 \quad \Rightarrow \ A = \$8400e^{0.0818(2.5)} = \$10,306 \]

It is better to open a savings account.

4. The present value of the \$10,000 payment is \$10,000, since no interest has been earned yet.

The \$7,000 amount (given 1 year later) has a present value of

\[ P = A(1 + \frac{r}{n})^{-nt} \]

\[ \text{where} \quad A = 7000, \quad r = 0.07, \quad t = 1 \]

\[ P = \$7000(1 + \frac{0.07}{1})^{-1(1)} = \$6526.76 \]

The \$4,000 amount (given 2 years later) has a present value of

\[ \$4000(1 + \frac{0.07}{1})^{-2(1)} = \$3477.43 \]

The total present value is 10,000 + 6526.76 + 3477.43 = \$20,004.19.

5. In each case, compare present value of all payments (with maximum legal interest rate) with initial \$1000.

(a) 400 after 2, 4, 6 months (\( \frac{1}{6}, \ \frac{1}{3}, \ \frac{1}{2} \) year = \( t \))

\[ \text{Total PV} = 400e^{-0.20(\frac{1}{6})} + 400e^{-0.20(\frac{1}{3})} + 400e^{-0.20(\frac{1}{2})} \]

\[ = 400[0.96722 + 0.935507 + 0.904837] = \$1123.02 \quad \text{Illegal} \]

(b) 350 after 2, 4, 6 months (\( \frac{1}{6}, \ \frac{1}{3}, \ \frac{1}{2} \) year = \( t \))

\[ \text{Total PV} = 350e^{-0.20(\frac{1}{6})} + 350e^{-0.20(\frac{1}{3})} + 350e^{-0.20(\frac{1}{2})} \]

\[ = \$982.65 \quad \text{Legal} \]

(c) \$333.33 to be paid in 2, 4, 6 years.

\[ \text{Total PV} = 333.33[e^{-0.20(2)} + e^{-0.20(4)} + e^{-0.20(6)}] = \$973.61 \quad \text{Legal} \]

7. \[ 1 + (1.06)^1 + \cdots + (1.06)^{10} = \frac{1 - (1.06)^{11}}{1 - 1.06} = 14.9716 \]

12. \[ (e^{-0.65})^2 + \cdots + (e^{-0.65})^{40} = (e^{-0.65})^2[1 + e^{-0.65} + \cdots + (e^{-0.65})^{38}] \]

\[ = (e^{-0.65})^2 \left[ \frac{1 - (e^{-0.65})^{39}}{1 - e^{-0.65}} \right] \quad \text{where} \quad e^{-0.65} = 0.522045 \]

\[ = 0.5702 \]

19. Invested funds at 6% compounded annually. Future value of \$P\ now is \( A = P(1 + \frac{r}{N})^{Nt} = P(1.06)^t \), \( t \) in years, since \( N = 1, \ r = 0.06 \).

<table>
<thead>
<tr>
<th>Payment No.</th>
<th>( t ) = time held</th>
<th>Value at withdrawal</th>
</tr>
</thead>
<tbody>
<tr>
<td>Age 5: 1</td>
<td>12</td>
<td>( A(1.06)^{12} )</td>
</tr>
<tr>
<td>Age 6: 2</td>
<td>11</td>
<td>( A(1.06)^{11} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( \vdots )</td>
</tr>
<tr>
<td>Age 15: 11</td>
<td>2</td>
<td>( A(1.06)^2 )</td>
</tr>
<tr>
<td>Age 16: 12</td>
<td>2</td>
<td>( A(1.06)^1 )</td>
</tr>
</tbody>
</table>
Thus

\[
\text{Total at withdrawal} = A[1.06 + (1.06)^2 + \cdots + (1.06)^{12}]
\]

\[
= A(1.06)[1 + (1.06)^1 + \cdots + (1.06)^{11}]
\]

\[
= A(1.06) \frac{1 - (1.06)^{12}}{1 - 1.06}
\]

\[
= A(1.06)[16.86994] = A(17.882138)
\]

Since we want to have total = 20,000, \( a \) should be

\[
\frac{20,000}{17.882138} = 1118.44.
\]

<table>
<thead>
<tr>
<th>Payment No.</th>
<th>( t ) = time of payment (yrs)</th>
<th>Present Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5 = ( r/2 )</td>
<td>( 160(e^{-0.064})^{1/2} = 160\left(e^{-0.064}\right)^{1/2} ) = ( 160\left(e^{-0.032}\right)^{1} )</td>
</tr>
<tr>
<td>2</td>
<td>1.0 = ( r/2 )</td>
<td>( 160(e^{-0.064})^{2/2} = 160\left(e^{-0.064}\right)^{2/2} ) = ( 160\left(e^{-0.032}\right)^{2} )</td>
</tr>
<tr>
<td>3</td>
<td>1.5 = ( r/2 )</td>
<td>( 160(e^{-0.064})^{3/2} = 160\left(e^{-0.064}\right)^{3/2} ) = ( 160\left(e^{-0.032}\right)^{3} )</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>19</td>
<td>9.5 = ( r/2 )</td>
<td>( 160(e^{-0.064})^{19/2} = 160\left(e^{-0.064}\right)^{19/2} ) = ( 160\left(e^{-0.032}\right)^{19} )</td>
</tr>
<tr>
<td>20</td>
<td>10.0 = ( r/2 )</td>
<td>( 160(e^{-0.064})^{20/2} = 160\left(e^{-0.064}\right)^{20/2} ) = ( 160\left(e^{-0.032}\right)^{20} )</td>
</tr>
<tr>
<td>Bond redeemed</td>
<td>10.0</td>
<td>( 5000(e^{-0.064})^{10} = 2636.46 )</td>
</tr>
</tbody>
</table>

Taking \( x = (e^{-0.064})^{1/2} = (e^{-0.032}) = 0.968501 \) in the geometric series formula, the Total Present Value is (PV of face value) + (PV of coupons), so:

\[
\text{Total PV} = 2636.46 + 160\left(e^{-0.032}\right)^{1} + (e^{-0.032})^{2} + \cdots + (e^{-0.032})^{20}\]

\[
= 2636.46 + 160\left(e^{-0.032}\right)^{1} + (e^{-0.032})^{1} + \cdots + (e^{-0.032})^{19}\]

\[
= 2636.46 + 160\left(e^{-0.032}\right)^{1} \left[1 - (e^{-0.032})^{20}\right] \]

\[
= 2636.46 + 160(0.968507)\left[15.009726\right] = 2636.46 + 2325.92 = \$4962.38
\]

Parts (i) and (ii) are handled similarly.

26 If “cost of money” is 6% compounded annually (\( N = 1 \)), a future payment of \( A \) dollars is discounted to \( A(1 + \frac{r}{N})^{-Nt} = A(1.06)^{-t} \) dollars. If the cost of money is 3.25%, the discounted value would be \( A(1.0325)^{-t} \), which is larger; under this scheme the project seems more profitable. There are various reasons, pro and con, for using the unrealistic rate. One is that someone wants to build the dam and fudges the figures to make it look better. Another is that if inflation is taken into account, 3.25% is not unrealistic if we compute in “constant dollars.”

Section 4.4

3 \( Q(t) = 0.71C = Ce^{-(1.212 \times 10^{-4})t} \) (Note that \( C \) cancels from both sides.)

\[
\ln 0.71 = \ln e^{-(1.212 \times 10^{-4})t} = -(1.212 \times 10^{-4})t
\]

\[
t = \frac{\ln 0.71}{-1.212 \times 10^{-4}} = \frac{\ln 0.71}{-1.212} \times 10^{4} = 2826 \text{ years}
\]

5 \( Q(t) = 0.78C = Ce^{-(0.154 \times 10^{-9})t} \)

\[
\ln 0.78 = \ln e^{-(0.154 \times 10^{-9})t} = -(0.154 \times 10^{-9})t
\]

\[
t = \frac{\ln 0.78}{-0.154 \times 10^{-9}} = 1.6 \times 10^{9} \text{ or 1 billion, 600 million years}
\]

(ii) \( t = \frac{\ln 0.85}{-0.154 \times 10^{-9}} = 1.1 \times 10^{9} \) years
(iii) \( t = \frac{\ln 0.92}{-0.154 \times 10^{-9}} = 5.4 \times 10^8 \) years

7. \( t = 2.598 \) days. We must find \( k \) before determining the half-life. \( 0.80C = Ce^{kt} \implies \ln 0.8 = \ln e^{kt} = kt \implies k = \frac{\ln 0.8}{2.598} = -0.086. \) To get the half-life \( T: \)

\[
Q(t) = \frac{1}{2} C = C e^{-0.086T} \\
\ln \frac{1}{2} = \ln e^{-0.086T} = -0.086T \\
T = \frac{\ln 1/2}{-0.086} = 8.06 \text{ days}
\]

9. \( 3C = Ce^{k \cdot 35}, \) \( t = 35 \) years. So \( k = \frac{\ln 3}{35} = 0.031. \) In the year 2000, \( N(2000) = N(1940)e^{(0.031)60} = 2 \times 10^7 e^{(0.031)60} = 1.3 \times 10^8. \)

15. \( I = I_0 e^{-0.5x}, \) \( \ln \left( \frac{I}{I_0} \right) = \ln e^{-0.5x} \implies x = -2 \ln(I/I_0) \)

(i) \( \frac{I}{I_0} = 0.5 \implies x = -2 \ln(0.5) = 1.4 \) feet, the depth of water at which the intensity is half of what it was at the surface. (ii) \( \frac{I}{I_0} = 0.7 \implies x = -2 \ln(0.7) = 0.7 \) feet (iii) \( \frac{I}{I_0} = 0.9 \implies x = -2 \ln(0.9) = 0.2 \) feet