Reconstructing Volatility

New techniques for understanding the implied volatility of multi-asset options

Speaker: Marco Avellaneda

Avellaneda, Boyer-Olson, Busca and Friz: `Reconstructing Volatility', RISK Oct 2002; `Large Deviations Methods and the Pricing of Index Options in Finance', CRAS Paris 2003

Outline

• Major US indices and ETFs
• Implied volatility surfaces of single stocks and indices
• Marginalization
• ‘Reconstructing’ the implied volatility of index options
• Steepest-descent Approximation
• The most likely market configurations
• Multivariate stochastic volatility models
• Moment-matching technique: Lee, Wang and Karim
• Cross-currency options
U.S. Equities: Main Sectors & Their Indices

- Major Indices: SPX, DJX, NDX
  SPY, DIA, QQQ (Exchange-Traded Funds)

- Sector Indices & Index Trackers:
  Semiconductors: SMH, SOX
  Biotech: BBH, BTK
  Pharmaceuticals: PPH, DRG
  Financials: BKX, XBD, XLF, RKH
  Oil & Gas: XNG, XOI, OSX
  High Tech, WWW, Boxes: MSH, HHH, XBD, XCI
  Retail: RTH

All these indices have options

Trading statistics (AMEX)

AVERAGE TRADED VOLUME IN DOLLARS LAST MONTH

<table>
<thead>
<tr>
<th>Index</th>
<th>Volume (MM USD)</th>
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<tr>
<td>DIA</td>
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<td>QQQ</td>
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<tr>
<td>SPY</td>
<td>4000</td>
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Components of NASDAQ 100 Trust (AMEX:QQQ)

- Capitalization-weighted average of 100 largest stocks in NASDAQ
- QQQ trades as a stock
- QQQ index options are the most heavily traded contract in AMEX
### Morgan Stanley 35 (MSH)

- **35 Underlying Stocks**
- Equal-dollar weighted index, adjusted annually
- Each stock has typically O(30) options over a 1yr horizon

<table>
<thead>
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<th>Underlying Stocks</th>
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<tbody>
<tr>
<td>ADP</td>
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<td>XLNX</td>
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### SOX, XNG, XOI

- **SOX**
  - ALTR
  - AMAT
  - AMD
  - INTC
  - KLAC
  - LLTC
  - LSIX
  - LSI
  - MOT
  - MU
  - NSM
  - NVLS
  - RMBS
  - TER
  - TXN
  - XLNX

- **XNG**
  - APA
  - APC
  - BR
  - BRR
  - EEX
  - ENE
  - EOG
  - EPG
  - KMI
  - NBL
  - NFG
  - OEI
  - PPP
  - STR
  - WMB

- **XOI**
  - AHC
  - BP
  - CHV
  - COC.B
  - XOM
  - KMG
  - OXY
  - P
  - REP
  - RD
  - SUN
  - TX
  - TOT
  - UCL
  - MRO

... & many others
Example: Basket of 20 Biotechnology Stocks (Components of BBH)

<table>
<thead>
<tr>
<th>Ticker</th>
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<th>ATM ImVol</th>
<th>Ticker</th>
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<td>BBH</td>
<td>-</td>
<td>32</td>
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AOL Jan 2001 Options:
Implied volatility curve on Dec 20, 2000
Market close

Stock probability is not lognormal
The AOL "volatility skew" for several expiration dates

BBH March 2003 Implied Vols
Pricing Date: Jan 22 03 10:42 AM
Implied Volatility Curve for Options on Dow Jones Average

Stylized facts about equity volatility curves

- Implied volatility curves are typically downward sloping
- Counterexamples: precious metal stocks are upward sloping
- There is little curvature (or smile). Skew is important.
Modeling single-stock volatility curves

\[ \frac{dS}{S} = \sigma_i dW \]

Price dynamics not lognormal

\[ \sigma_i = \sigma(S, t) \]

Dupire's Local Volatility

\[ \frac{d\sigma_t}{\sigma_t} = \kappa dZ, \]

Stochastic Volatility

\[ \sigma_{\text{implied}}(K, T) = \sigma_{\text{implied}}(S, T) \cdot (1 + a \ln(K / S)) \]

Empirical Fit

Jumps …

What is the relation between index options and options on the components?

Standard (log-normal) Volatility Formula for Index Options

\[ \sigma_i^2 = \sum_{j=1}^{N} p_j^2 \sigma_j^2 + \sum_{i \neq j} p_i p_j \sigma_i \sigma_j \rho_{ij} \quad (*) \]

Does not apply when volatilities are strike-dependent

How can we incorporate volatility skew information into (*)?
“Marginalization”

Consider an \( n+1 \) dimensional diffusion:

\[
\begin{align*}
    (X,Y) & \in \mathbb{R}^{n+1} \\
    dX^i & = \sum_{j=1}^{n} \sigma^j_i(X,Y,t)dW^j + \mu^i(X,Y,t)dt \quad i = 1, \ldots, n, \\
    dY & = \sum_{j=1}^{n} \kappa^j_j(X,Y,t)dW^j + \nu(X,Y,t)dt
\end{align*}
\]

Problem: given a starting point, \((X_0, Y_0, t_0)\), find a one-dimensional diffusion

\[
d\bar{Y} = \bar{\kappa}(\bar{Y}, t)dW + \bar{\nu}(\bar{Y}, t)dt
\]

such that \( Y(t) \) and \( \bar{Y}(t) \) have the same probability distributions for all \( t > t_0 \).

Fokker-Planck Equation

\[
\pi(x, y, t) \equiv \pi(X_0, Y_0, t_0; x, y, t)
\]

transition probability function

\[
\frac{\partial \pi(x, y, t)}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial y^2} \left( \kappa^2 \pi(x, y, t) \right) + \sum_{i} \frac{\partial^2}{\partial y \partial x_i} (\ldots) + \\
\sum_{ij} \frac{\partial^2}{\partial x_i \partial x_j} (\ldots) - \frac{\partial}{\partial y} (\nu \pi(x, y, t)) - \sum_{i} \frac{\partial}{\partial x_i} (\ldots)
\]
Computing the marginal distributions…

\[ \pi(y,t) \equiv \int \pi(x,y,t) dx \]

\[ \frac{\partial \pi(y,t)}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial y^2} \left( \kappa^2 \pi(\cdot,y,t) \right) - \frac{\partial}{\partial y} \left( \nu \pi(\cdot,y,t) \right) \]

General formula

\[ \frac{\partial \pi(y,t)}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial y^2} \left( \frac{\kappa^2 \pi(\cdot,y,t)}{\pi(y,t)} \pi(y,t) \right) - \frac{\partial}{\partial y} \left( \frac{\nu \pi(\cdot,y,t)}{\pi(y,t)} \pi(y,t) \right) \]

integration with respect to x eliminates terms with x-derivatives

\[ \text{new 1D diffusion diffusion equation} \]

Equivalent 1-D diffusion

\[ \kappa^2(y,t) \equiv \frac{\kappa^2(\cdot,y,t)\pi(\cdot,y,t)}{\pi(y,t)} = \int \frac{\kappa^2(x,y,t)\pi(x,y,t)dx}{\pi(x,y,t)dx} \]

\[ \nu(y,t) \equiv \frac{\nu(\cdot,y,t)\pi(\cdot,y,t)}{\pi(y,t)} = \int \frac{\nu(x,y,t)\pi(x,y,t)dx}{\pi(x,y,t)dx} \]

\[ \kappa^2(y,t) = E\{e^{2(X(t),Y(t),t)|Y(t)=y}\} \]

\[ \nu(y,t) = E\{e^{\nu(X(t),Y(t),t)|Y(t)=y}\} \]

Equivalent parameters = conditional expectations of local parameters
Application to Index Options

\[ I = \sum_{i=1}^{n} w_i S_i \quad \text{Index = weighted sum of stock prices (constant weights)} \]

Diffusion eq. for each stock reflects vol skew

\[
\begin{align*}
\frac{dS_i}{S_i} &= \sigma_i(S_i,t) dW_i + \mu_i dt, \\
\mu_i &= r - d_i, \\
E(dW_i dW_j) &= \rho_{ij} dt
\end{align*}
\]

\[
\frac{dI}{I} = \sigma_{loc}(S,t)dZ + \mu_{loc}(S,t)dt
\]

\[
\sigma^2_{loc}(S,t) = \frac{\sum_{i,j} \sigma_i(S_i,t) \sigma_j(S_j,t) w_i w_j S_i S_j \rho_{ij}}{I^2}
\]

\[
\mu_{loc}(S,t) = \frac{\sum \mu_i w_i S_i}{I}
\]

Characterization of the equivalent volatility for the index

\[
\varphi^2(I,t) = E \left\{ \frac{\sum_{i,j} \sigma_i(S_i,t) \sigma_j(S_j,t) S_i(t) S_j(t) w_i w_j \rho_{ij}}{I^2} \mid \sum_i w_i S_i(t) = I \right\}
\]

\[
= E \left\{ \sum_{i,j} p_i(S(t)) p_j(S(t)) \sigma_i(S_i(t) \sigma_j(S_j(t) \sigma_i(S_i(t) \sigma_j(S_j(t) \rho_{ij} \mid \sum_i w_i S_i(t) = I \right\}
\]

\[
p_i(S) = \frac{w_i S_i}{\sum_j w_j S_j}, \quad i = 1, \ldots, n.
\]

* sigma_loc can be seen as a 'stochastic vol' driving the index

* sigma_bar is then the 'averaged vol'
Varadhan’s Formula

\[ dX_i = \sum_{j=1}^{n} \sigma_{ij}(X,t)dW_j, \quad E[dW_i, dW_j] = \rho_{ij}dt \]

\[ X_i(0) = x_i \]

\[ \log \text{Prob.}\{ X(t) = y | X(0) = x \} \approx -\frac{d^2(x,y)}{2t}, \quad (\sigma^2)^t \approx 1 \]

\[ d^2(x,y) = \inf_{y(0)=x,y(t)=y} \int_0^t g(y(s)) \gamma'(s) \gamma'(s)ds \]

\[ g(x) = a^{-1}(x), \quad a_g(x) = \sigma_i(x,0)\sigma_j(x,0)\rho_{ij} \]

In practice: dimensionless time \( \sim 0.02 \)

Steepest-descent approximation

Change to log-scale: \( x_i = \log \left( \frac{S_i}{S_i(0)e^{\mu r}} \right) = \log \left( \frac{S_i}{F_i(t)} \right) \quad i = 1, 2, \ldots, n. \)

Formally,

\[ \sigma^2(I,t) = \frac{E[\sigma_{loc}^2(I(t)-I)]}{E[\delta(I(t)-I)]} \]

Applying Varadhan’s Formula,

\[ \sigma^2(I,t) \approx \sigma^2_{loc}(S^*, t) \quad S^*_i = S_i(0)e^{\mu r}e^{x_i^*} \]

where

\[ x^* = \arg \min \left\{ d^2(0,x) \sum_i w_i S_i(0)e^{\mu r}e^{x_i} = I \right\} \]
Steepest Descent = Most Likely Stock Price Configuration

Replace conditional distribution by “Dirac function” at most likely configuration

Characterization of MLC

Euler-Lagrange equations: find \( (x^*, \lambda) \) such that

\[
\int_0^{x^*_i} \frac{du}{\sigma_j(u)} = \lambda \sum_{j=1}^n p_j(x^*) \sigma_j(x^*_j) \rho_{ij} \quad i = 1, \ldots, n
\]

\[
\sum_{i=1}^n w_i S_i(0) e^{x^*_i + \mu_i t} = I
\]

\[
\bar{\sigma}^2(1,t) = \sum_{i,j=1}^n p_i(x^*) p_j(x^*) \sigma_i(x^*_i) \sigma_j(x^*_j) \rho_{ij}
\]
Solution of linearized system in terms of the stock betas

\[ \sigma_i^\prime(0) = \sum_{j=1}^n \rho_i \rho_j \sigma_i(0) \sigma_j(0) \rho_{ij} \]

\[ \bar{x} = \ln \left( \frac{I(0)e^x}{I(0)} \right) \]

\[ x_i^* = \frac{\bar{x}}{\sigma_i^\prime(0)} \sum_{j=1}^n \rho_i \rho_j \sigma_j(0) \sigma_i(0) = \frac{\bar{x}}{\sigma_i^\prime(0)} \text{Cov}(x_i, \bar{x}) \]

\[ x_i^* = \beta \bar{x} \]

\[ \hat{\beta} = \text{Cov} \left( \frac{\Delta S}{S}, \frac{\Delta I}{I} \right) \left[ \text{Var} \left( \frac{\Delta I}{I} \right) \right] \]

Most likely config. : described by the risk-neutral regression coefficients of stock returns with the index return ("micro" CAPM)

Expression in terms of Implied Volatilities and Black-Scholes Deltas

- Seek direct relation between implied volatilities of single-stock options and implied volatility of index options
- Tool: Berestycki-Busca-Florent large-deviations result for single-stock ("1/2 slope rule")

\[ \sigma^{\text{impl}}(x) = \left( \frac{1}{x} \int_0^e \frac{du}{\sigma(u)} \right)^{-1} \]

\[ \sigma^{\text{impl}}(x) = \frac{1}{2} \left( \sigma^{\text{impl}}(0) + \sigma(x) \right) \]
The Formula In Terms of Black Scholes Deltas

From Euler-Lagrange equations:

\[
\frac{\ln(S' / F)}{\sigma_{\text{impl}}(S')} = \frac{\ln(I / I_j)}{\sigma_{\text{impl}}(I)} \times \sum_{j=1}^{n} \rho_j p_j \left( \frac{\sigma_{\text{impl}}(F)}{\sigma_{\text{impl}}(I_j)} \right) = \ln\left( \frac{I / I_j}{I / I_j} \right) \text{Corr}\left( \frac{\Delta S_j}{S_j}, \frac{\Delta I}{I} \right)
\]

Translate into Black-Scholes deltas

\[
\Delta \equiv N\left( \frac{1}{\sigma \sqrt{t}} \ln\left( \frac{F}{S'} \right) + \frac{1}{2} \sigma \sqrt{t} \right), \quad N(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{z^2}{2}} dz
\]

\[
\Delta_i = N^{-1}(\Delta_i) \times \sum_{j=1}^{n} \rho_j p_j \left( \frac{\sigma_{\text{impl}}(F)}{\sigma_{\text{impl}}(I_j)} \right)
\]

Delta of each stock is a function of delta of index option

Reconstruction Formula for the Index Volatility

\[
\sigma_{\text{impl}}(\Delta_i) = \frac{\sigma_{\text{impl}}(0.5) + \bar{\sigma}(\Delta_i)}{2}
\]

\[
\bar{\sigma}(\Delta_i) = \sqrt{\sum_{i=1}^{n} \rho_j p_i \left( 2\sigma_{\text{impl}}(\Delta_i) - \sigma_{\text{impl}}(0.5) \right) \left( 2\sigma_{\text{impl}}(\Delta_i) - \sigma_{\text{impl}}(0.5) \right) / (2\sigma_{\text{impl}}(\Delta_i) - \sigma_{\text{impl}}(0.5))}
\]

\[
p_i = \text{capitalization of stock in index (%)}
\]
**Correlation & Optimal Configurations**

- If the stocks are perfectly correlated,
  \[
  \sum_{j=1}^{n} p_j \sigma_{j}^{\text{impl}}(F_j) = \sigma_{i}^{\text{impl}}(I_f) \quad \therefore \quad \Delta_i = \Delta_f
  \]

- If the stocks are uncorrelated,
  \[
  \frac{\ln(S_i^f / F_i)}{\sigma_{i}^{\text{impl}}(S_i^f)} \approx \frac{\ln(I / I_f)}{\sigma_{i}^{\text{impl}}(I)} \quad \forall i
  \]  
  "Equal-delta approximation"

Less volatile & small stocks have smaller deltas closer to 50% (their skew does not affect the index as much)

**DJX: Dow Jones Industrial Average**

T=1 month

**DJX Nov 02**  Pricing Date: 10/25/02

- BidVol
- AskVol
- SDA
T = 2 months

DJX Dec 02 Pricing Date: 10/25/02

T = 3 months

DJX Jan 03 Pricing Date: 10/25/02
T = 5 months

DJX Mar 03    Pricing Date: 10/25/02

T = 7 months

DJX June 03    Pricing Date: 10/25/02
T = 6 months

BBH Apr 03 Date: Oct 25 02

Vol

Delta

Is dimensionless time is too long? (Error bars: Juyoung Lim)
Is correlation causing the discrepancy?

S&P 100 Index Options
(Quote date: Aug 20, 2002)

Expiration: Sep 02
S&P 100 Index Options
(Quote date: Aug 20, 2002)

Expiration: Dec 02

General Stochastic Volatility Systems

\[
\frac{dS_i}{S_i} = \sigma_i dW_i \quad \frac{d\sigma_i}{\sigma_i} = \kappa_i dZ_i
\]

\[E(dW_i dW_j) = \rho_{ij} dt \quad E(dW_i dZ_j) = \kappa_{ij} dt\]

\[\frac{dx_i}{I} = y_i \quad x_i = \frac{dS_i}{S_i} \quad y_i = \frac{d\sigma_i}{\sigma_i}\]

Look for most likely configuration of stocks and vols \(\{x_1, \ldots, x_n, y_1, \ldots, y_n\}\) corresponding to a given index displacement \(x\).
Most likely configuration for Stochastic Volatility Systems

\[ x^*_i = \beta_i \bar{x} \]

\[ \beta_i = \frac{\sigma_i \rho_d}{\sigma_l} \]

Most likely configuration for stocks moves and volatility moves, given the index move

\[ y^*_i = \gamma_i \bar{x} \]

\[ \gamma_i = \frac{k_i \tau_i}{\sigma_l} \]

\[ \sigma^2_i(\bar{x}, t) \equiv \sum_{ij=1}^n p_i p_j \sigma_i (0, t) \sigma_j (0, t) e^{\gamma_i \bar{x}} e^{\gamma_j \bar{x}} \rho_{ij} \]

Numerical Example

\[ n = 2 \quad p_1 = p_2 = 0.5 \]

\[ \sigma_1 = 20\% \quad \gamma_1 = -1 \quad \sigma_2 = 30\% \quad \gamma_2 = -0.5 \]

\[ \rho = 40\% \]
Method I: Dupire & Most Likely Configuration for Stock Moves

N-dimensional Equity market

\[ \sigma_1(x_1, t) \]

\[ \sigma_i(x_i, t) \]

\[ \sigma_n(x_n, t) \]

• Step 1: Local volatility for each stock
• Step 2: Find most likely configuration for stocks

Method II: Stochastic Volatility System and joint MLC for Stocks and Volatilities

N-dimensional Equity market

\[ \sigma_i(x, t) \]

• Only one step: compute the most likely configuration of stocks and volatilities at the same time
Are methods I and II "equivalent"?

Answer is NO, in general.

\[ \sigma_i(x, t) = \sigma_i(0, t) e^{\omega_i x_i} \quad \sigma_i = \frac{\kappa_i r_i}{\sigma_i} \]

Index vol., Method I

\[ \sigma_i^2(x_i, t) = \sum_{ij} p_i p_j \sigma_i(0, t) \sigma_j(0, t) \rho_{ij} e^{-\kappa_i t_i} e^{\sigma_i \beta_i t_i} \]

Index vol., Method II

\[ \sigma_i^2(x_i, t) = \sum_{ij} p_i p_j \sigma_i(0, t) \sigma_j(0, t) \rho_{ij} e^{-\gamma_j t_i} e^{\gamma_j \beta_j t_i} \]

Dupire local vol. for single names

Equivalence holds only under additional assumptions on stock-volatility correlations

\[ \sigma_i \beta_i = \frac{\kappa_i r_{ii} \sigma_i \rho_{ii} \sigma_i}{\sigma_i} = \frac{\kappa_i r_{ii} \rho_{ii} \sigma_i}{\sigma_i} \]

Method I

\[ \gamma_i = \frac{\kappa_i r_{ii} \sigma_i}{\sigma_i} \]

Method II

\[ r_{ii} = r_{ii} \rho_{ii} \]

Conditions under which both methods give equivalent valuations
The corresponding stock-volatility correlation structure is `diagonal'

\[ dz_i = \sum_k m_{ik} dW_k + \alpha_i d\xi_i \quad E(dW_k, d\xi_l) = 0 \]

\[ \therefore r_{ij} = \sum_k m_{ik} \rho_{kj} \]

\[ r_{ij} = r_{ii} \rho_{ij} \quad \Rightarrow \quad dz_i = r_{ii} dW_i + \alpha_i d\xi_i \]

Examples: Dupire, CEV, uncoupled SV mode

**Numerical Example**

\[ \sigma_1 = 20\%, \sigma_2 = 30\%, \rho = 40\% \]

\[ r = \begin{bmatrix} -0.7 & -0.5 \\ -0.6 & -0.7 \end{bmatrix}, \quad \kappa_1 = \kappa_2 = 50\% \]

![Graph showing IVOL (%)](image)
Lee, Wang and Karim

Gram-Charlier expansion  

\[ x = \log \left( \frac{K}{F} \right), \quad \sigma^2 = \text{Var}(x), \quad s = \frac{E(x^3)}{\sigma^3}, \quad k = \frac{E(x^4)}{\sigma^4} - 3 \]

\[ \sigma_{\text{am}} = \sigma \left[ 1 + \frac{s}{3!} \sigma \sqrt{T} + \frac{k}{4!} \left( \frac{7}{4} \sigma^2 T - 1 \right) \right] \]

\[ \text{Skew} = \frac{1}{\sqrt{T}} \left( \frac{s}{3!} + \frac{2k \sigma \sqrt{T}}{4!} \right), \quad \text{Smile} = \frac{1}{T} \frac{k}{4! \sigma} \]

\[ \sigma(x) = \sigma_{\text{am}} + x \cdot \text{Skew} + x^2 \cdot \text{Smile} \]

Calculate Index Moments

\[ E(x^2) = \sum_{ij} p_i p_j \sigma_i \sigma_j \rho_{ij} \]

\[ E(x^3) = \sum_{ijk} p_i p_j p_k \sigma_i \sigma_j \sigma_k E(z_i z_j z_k) \]

\[ E(x^4) = \sum_{ijk} p_i p_j p_k \sigma_i \sigma_j \sigma_k \sigma_i E(z_i z_j z_k z_l) \]

``Gaussian’’ closure for third and 4th moments
3rd Moments

\[ E(z_i z_j z_k) = \text{Sym}[\text{Corr}(z_i, z_k) \text{Corr}(z_i, z_k) s_i] = \text{Sym}[\rho_{ik} s_i] \quad \text{if } i \neq j \neq k \]

\[ E(z_i z_j z_k) = \text{Corr}(z_j, z_k) E(z_i^3) = \rho_{ik} s_i \]

\[ E(z_i z_k) = E(z_i^3) = s_i \]

\[ s_i = \frac{1}{\sigma_k} \sum_{qk} p_j p_k \sigma_j \sigma_k \rho_{jq} \rho_{ik} s_i \]

Consistent with the SDA approximation formula!

Stochastic Correlation Model

- Lee, Wang and Karim: stochastic correlation
- Linear fit

\[ \bar{\rho} = \alpha + \beta \ln I + \epsilon \]

Rho_bar is the 'average' correlation

\[ OEX : \quad \beta = -0.66 \]

\[ BXX : \quad \beta = -0.34 \]

This model can be used to improve the accuracy of the SDA.
Cross-Currency Options

Major currency triad: USD/JPY, USD/EUR, JPY/EUR

\[ S_1 = \text{JPY/USD} \quad x_1 = \ln \frac{S_1}{S_1(0)e^{\mu_1t}} \]

\[ S_2 = \text{EUR/USD} \quad x_2 = \ln \frac{S_2}{S_2(0)e^{\mu_2t}} \]

\[ I = \text{EUR/JPY} \quad \bar{x} = \ln \frac{I}{I(0)e^{(\mu_1-\mu_2)t}} \]

Formally, the answer is the same as in the index case. The `geometry` is different.