Weighted Monte-Carlo Methods for Multi-Asset Equity Derivatives: Theory and Practice

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Summary

- Statement of the Calibration Problem for Multi-Asset Equity Derivatives
- Weighted Monte Carlo simulation (max-entropy)
- Application to Arbitrage Pricing of Basket Options
- Comparison between WMC and Steepest Descent Method
- Comments on Correlation Skew and the statistics of Implied and Historical Correlations
Calibration Problem for Multi-Asset Equity Derivatives

Given a group, or collection of stocks, build a stochastic model for the joint evolution of the stocks with the following properties:

• The associated probability measure on market scenarios is risk-neutral: all traded securities are correctly priced by discounting cash-flows

• The associated probability measure is such that stock prices, adjusted for interest and dividends, are martingales (local risk-neutrality)

• The model simulates the joint evolution of ~ 100 stocks

• All options (with reasonable OI), forward prices, on all stocks, must be fitted to the model. Number of constraints ~500 to ~1000 or more

• Efficient calibration, pricing and sensitivity analysis in real-time environment

Example: Basket of 20 Biotechnology Stocks (Components of BBH)

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<th>ATM ImVol</th>
<th>Ticker</th>
<th>Price</th>
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Multi-Dimensional Diffusion Model

\[ \frac{dS_i}{S_i} = \sigma_i dZ_i + \mu_i dt \quad \mu_i = r - d_i \quad \text{ensures martingale property} \]

\[ dZ_i = \text{Brownian motion increment} \]

\[ E(dZ_idZ_j) = \rho_{ij} dt \]

1-Dimensional Problems
- Dupire: local volatility as a function of stock price \( \sigma = \sigma(S,t) \)
- Hull-White, Heston: more factors to model stochastic volatility
- Rubinstein, Derman-Kani: implied "trees"

These methods do not generalize to higher dimensions!
Main Challenges in Multi-Asset Models

- Modeling correlation, or co-movement of many assets
- Correlation may have to match market prices if index options are used as price inputs (time-dependence)
- Fitting single-asset implied volatilities which are time- and strike-dependent
- Large body of literature on 1-D models, but much less is known on intertemporal multi-asset pricing models

Beware of "magic fixes", e.g. Copulas

Weighted Monte Carlo

Avellaneda, Buff, Friedman, Grandchamp, Kruk: IJTAF 1999

- Build a discrete-time, multidimensional process for the asset price
- Generate many scenarios for the process by Monte Carlo Simulation
- Fit all price constraints using a Maximum-Entropy algorithm
Example 1: Discrete-Time Multidimensional Markov Process

Modeled after a diffusion

\[ S_{n+1}^{(i)} = S_n^{(i)} \left[ 1 + \sigma_n^{(i)} \left( \sum_{j=1}^{N} \alpha_{j,i} \xi_{n,j} \right) \sqrt{\Delta t} + \mu_n^{(i)} \Delta t \right] \]

\[ \xi_{n,j} = \text{i.i.d. normals} \]

- Correlations estimated from econometric analysis
- Vols are ATM implied or estimated from data
- Time-dependence, seasonality effects, can be incorporated

Example 2: Multidimensional Resampling

Bootstrap: B. Efron

\[ S_{ni} = \text{historical data matrix} \quad n \leq \nu \text{ (sample size)} \]

\[ X_{ni} = \frac{S_{ni} - S_{(n-1)i}}{S_{(n-1)i}} \quad Y_{ni} = \frac{X_{ni}}{\sqrt{\sum_{m=1}^{\nu} \left( X_{mi} - \bar{X}_i \right)^2}} \]

Use resampled standardized moves to generate scenarios

\[ S_{n+1}^{(i)} = S_n^{(i)} \left[ 1 + \sigma_n^{(i)} Y_{R(n),i} \sqrt{\Delta t} + \mu_n^{(i)} \Delta t \right] \]

\[ R(n) = \text{random number between 1 and } \nu \]

\( R(n) \) can be uniform or have temporal correlation
Two draws from the empirical distribution (12/99-12/00)

Calibration to Option and Forward Prices

- Evaluate Discounted Payoffs of reference instruments along different paths
  \[ g_{ij} = e^{-rT} \max \left( S_{1,i}^{u_j} - K, 0 \right) \]
  \( i = 1, \ldots, N \) (number of simulated paths)
  \( j = 1, \ldots, M \) (number of reference instrument\s)
  \( C_j \) = midmarket price of \( j^{th} \) reference instrument

\[ \begin{pmatrix} C_1 \\ \vdots \\ C_M \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} & \ldots & g_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ g_{M1} & \ldots & \ldots & g_{MN} \end{pmatrix} \begin{pmatrix} p_1 \\ \vdots \\ p_N \end{pmatrix} \]

- Repricing condition
  \[ C_j = E^P(g_j(S)), \quad j = 1, 2, \ldots, M \]
Maximum-Entropy Algorithm

\[ H(p) = - \sum_{i=1}^{N} p_i \log p_i = -D(p \| u) \quad u = \left( \frac{1}{N}, \ldots, \frac{1}{N} \right) \]

Algorithm

\[ \max_p H(p) \quad \text{subject to price constraints} \]
\[ \min_p D(p \| u) \quad " \]


Calibrated Probabilities are Gibbs Measures

Lagrange multiplier approach for solving constrained optimization gives rise to M-parameter family of Gibbs-type probabilities

\[ p_i = p_i = \frac{1}{Z(\lambda)} \exp \left[ \sum_{j=1}^{M} \lambda_{ij} g_{ij} \right], \quad i = 1, 2, \ldots, N \]

\[ Z(\lambda) = \sum_{i=1}^{N} \exp \left[ \sum_{j=1}^{M} \lambda_{ij} g_{ij} \right] \]

Boltzmann-Gibbs partition function

Unknown parameters
Calibration Algorithm
How do we find the lambdas?

- Minimize in lambda

\[ W(\lambda) = \log Z(\lambda) - \sum_{j=1}^{M} \lambda_j C_j \]

- \( W \) is a convex function
- The minimum is unique, if it exists
- \( W \) is differentiable in C, lambda
- Use L-BFGS Quasi-Newton gradient-based optimization routine

Boltzmann-Gibbs formalism

\[ \frac{\partial W(\lambda)}{\partial \lambda_j} = E^{\rho_X}(G_j(X)) - C_j \quad \text{Gradient= difference between market px and model px} \]
\[ \frac{\partial^2 W(\lambda)}{\partial \lambda_j \partial \lambda_k} = E^{\rho_X}(G_j(X)G_k(X)) - C_j C_k = Cov^{\rho_X}(G_j(X), G_k(X)) \quad \text{Hessian=covariance of cash-flows under pricing measure} \]

Numerical optimization with known gradient & Hessian
Least-Squares Variant

\[ \chi^2 = \sum_{j=1}^{M} \left( \sum_{i=1}^{N} g_{ij} p_i - C_j \right)^2 = \sum_{j=1}^{M} \left( E^p (g_j (S)) - C_j \right)^2 \]

\[ \min_p \left\{ -H(p) + \frac{\chi^2}{2\varepsilon^2} \right\} \quad \text{Max entropy with least-squares constraint} \]

\[ \min_{\lambda} \left\{ \ln Z(\lambda) + \sum_{j=1}^{M} \lambda_j C_j + \frac{\varepsilon^2}{2} \sum_{j=1}^{M} \lambda^2_j \right\} \quad \text{Equivalent to adding quadratic term to objective function} \]

Sensitivity Analysis

\( h(X) = \text{payoff function of ``target security''} \)

\( E^{P_X} (h(X)) = \text{model value of} \quad " \)

\[ \frac{\partial E^{P_X} (h(X))}{\partial C_j} = \frac{\partial E^{P_X} (h(X))}{\partial \lambda_k} \frac{\partial \lambda_k}{\partial C_j} \]

\[ = \text{Cov}^{P_X} (h(X), g_k (X)) \left( \frac{\partial C}{\partial \lambda_k} \right)^{-1}_{ij} \]

\[ = \text{Cov}^{P_X} (h(X), g_k (X)) \left( \text{Cov}^{P_X} (g_s (X), g_s (X)) \right)^{-1}_{jk} \]
Price-Sensitivities = "Betas"

Solve LS problem:

\[
\min_{\beta, \alpha} \sum_{i=1}^{v} p_i \left( h(X_i) - \alpha - \sum_{j=1}^{M} \beta_j G_j(X_i) \right)^2
\]

\[
h(X) = \alpha + \sum_{j=1}^{M} \beta_j g_j(X) + \varepsilon(X)
\]

Uncorrelated to \(g_j(X)\)

Minimal Martingale Measure?

Avellaneda and Fisher, forthcoming 2003

- Boltzmann-Gibbs posterior measure with price constraints is not a local martingale

- Remedy: include additional constraints:

\[
g(S) = \psi(s_1, ..., s_n)(s_{r:n} - s_{s:n}) \quad \psi(s_1, ..., s_n) = \text{polynomial function}
\]

Martingale constraint: \(E^\nu(g(S)) = 0\) for all \(\psi\)

- Constrained Max-Entropy problem with martingale constraints:
  Follmer-Schweitzer MEM under constraints

- In practice, use only low-degree polynomials (deg=0 or deg=1)
Performance of the algorithm for equity derivatives

- 100+ Stocks easily implemented
- 6 months to one year horizon (~ 5 expirations)
- 1000+ options and forwards
- Calibration time: < 5 minutes on single-processor PIII with 900 MHZ
- Scales almost linearly with number of processors.

Allows for real-time implementation and for exploring new groups of stocks with a relatively small computational effort

Application: “Relative Value” Pricing of Index Options

- Observe current prices of index options
- Observe current prices of options on index components
- Build model to determine whether index options are “rich” or “cheap” in relation to the components

Crucial points:

(I) incorporate volatility skew for each component
(II) fit the information together (across components)
Some Exchange-Traded Funds and Indices with Options

QQQ: NASDAQ 100 Trust (AMEX)
MSH: Morgan Stanley 35 High-Technology Index (AMEX)
SOX: Semiconductor Index (PHLX)
BTK: Biotechnology Index (AMEX)
XNG: Natural Gas Index (AMEX)
XAU: Gold Index (AMEX)
NASDAQ 100 Index (NDX) and ETF (QQQ)

- Capitalization-weighted average of 100 largest stocks in NASDAQ
- QQQ trades as a stock
- QQQ index option is the most actively traded contract in AMEX

Morgan Stanley 35 (MSH)

- 35 Underlying Stocks
- Equal-dollar weighted index, adjusted annually
- Each stock has typically O(30) options over a 1yr horizon
Test problem: 35 tech stocks

Price options on basket of 35 stocks underlying the MSH index

Number of constraints: 876

Number of paths: 10,000 to 30,000 paths

Optimization technique: Quasi-Newton method (explicit gradient)
Near-month options
(Pricing Date: Dec 2000)

MSH Basket option: model vs. market
Front Month

Second-month options

Basket option: model vs. market
Third-month options

Basket option: model vs. market

Six-month options

Basket option: model vs. market
Entropy as a Measure of Information Content

Claude Shannon, 1941

\[ 0 \leq H(p) \equiv -\sum_{k=1}^{N} p_k \log p_k \leq \log N \]

\[ H(p) = \log N \Rightarrow p_k = 1/N \quad \text{No discrepancy between current market and prior distribution} \]

\[ H(p) \approx 0 \quad \Rightarrow \quad \text{Extreme departure from prior distribution: signals internal mispricing } \odot (\text{or data entry error } \odot) \]

Entropy score = \( 100 \cdot \frac{H(p)}{\log N} \quad \text{Measures "information content" of the data} \)

Evolution of Measured Entropy Score: DJX components

Industry-diversified blue-chip index

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<th>Entropy Score</th>
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Evolution of Measured Entropy Score: BBH Components

Biotechnology sector index

BBH Index Options

BBH Nov 02
Quote Date Oct 28
Comparison With Steepest-Descent Approximation

Avellaneda, Boyer-Olson, Busca, Friz: RISK 2002, CRAS Paris 2003

- Based on Steepest Descent Approximation, or short-time asymptotics for diffusion kernels
- Use implied volatilities (co-terminal) of options on underlying stocks
- Use historical or estimated correlation matrix

\[
\sigma_i(\Delta) \equiv \frac{1}{2} \sigma_{i,ATM} + \frac{1}{2} \sqrt{\sum_{j=1}^{M} p_j \rho_{ij} (2\sigma_j(\Delta_i) - \sigma_{i,ATM}) (2\sigma_j(\Delta_i) - \sigma_{j,ATM})}
\]

\[
\Delta_i = \left[ N^{-1}(\Delta) \times \sum_{j=1}^{M} p_j \rho_{ij} \frac{\sigma_{j,ATM}}{\sigma_{i,ATM}} \right]
\]

\[
N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy
\]
S&P 100 Index Options
(Quote date: Aug 20, 2002)

Expiration: Sep 02

Expiration: Oct 02
S&P 100 Index Options
(Quote date: Aug 20, 2002)

Expiration: Nov 02

S&P 100 Index Options
(Quote date: Aug 20, 2002)

Expiration: Dec 02
Dow Industrial Average (DJX)

Amex Biotechnology Index (BTK)
Conclusions

- Weighted Monte Carlo: convenient framework for pricing and hedging derivatives with many underlying assets
- Non-parametric: avoids the use of latent variables (‘market model’) and is able to fit econometric and price data in detail
- Results are consistent with Steepest Descent reconstruction algorithm for pricing index options
- Entropy ~ information content of the market. Allows for monitoring the information content in a large group of quotes and to search for arbitrage opportunities
- Obvious important extensions: credit derivatives and capital structure arbitrage (joint model for equity, corporate debt and credit derivatives)