From SABR to Geodesics

A systematic approach for modeling volatility curves with applications to option market-making and pricing multi-asset equity derivatives

The Importance of Having a Listed Derivatives Historical Database

- Equity derivatives analysts can now access historical databases on listed options at relatively low cost. They can:
  - Back-test models, especially calibration aspects
  - Debunk myths about option models (there are plenty of them!)
  - Back-test option strategies systematically, as is done for cash trading
  - Test the stability of a skew and vol surface model with real data
  - Learn more, by observation, get ideas…

Recommendation: IVY OptionMetrics
Fitting Volatility Skews

SPX JUN04 (PRICING DATE MAY 22)

Blue line = average implied vol (puts/calls)
Pink line = fitted parabola

Zoom into the region -0.2<ln(K/F)<0.2

If you zoom into the region of interest, the parabolic fit is seen as clearly inadequate.
Reason: the out-of-the-money options "lift" the curve.

Parabolic fits are not consistent with arbitrage-free pricing.
Parabolic fitting requires Delta Truncation!

Fit only volatilities such that -0.2 < x < 0.2

\[ \sigma_{\text{parabolic}}(x) = 0.16 - 0.34x + 4.45x^2 \]

\[ = 0.16 \times (1 - 2.1x + 27.33x^2) \]

Truncated Parabolic Fit: a look at the full curve

Out of the money options are not guaranteed to be well-fitted
Using a better spline to fit the data (from SABR)

\[
\sigma_{imp}(x) = \ln \left( \frac{1 - e^{-\beta x}}{\sigma_0 \beta} + \sqrt{1 + \kappa^2 \left( \frac{1 - e^{-\beta x}}{\sigma_0 \beta} \right)^2} \right) \approx -x \sigma_0 \beta 
\]

Sigma, beta and kappa are adjustable parameters

Formula is derived from a stochastic volatility model so it does not violate arbitrage conditions

Fitting a SABR-like spline to the SPX front-month curve
Evolution of the slope of the 30-day implied volatility curve, 1996-2004

Avellaneda & Lee, 2005
Evolution of ratio [slope/leverage coefficient]
The ``roaring 90's''!

\[ \beta / \gamma = \text{Avellaneda & Lee, 2005} \]

Differential Geometry and Implied Volatility Modeling

Avellaneda & Lee, 2005
Factor Models and Diffusion Kernels

CIR-type setting, $X=$ state variables
$W=$ m-dim Brownian motion

$$dX_i = \sum_{k=1}^{m} \sigma_{i,k}^2 dW_k + b_i dt, \quad i = 1, 2, \ldots, n$$

$$\pi(x, t; y, T) = \text{Prob}\{x(T) = y|x(t) = x\}$$

Fokker-Planck Equation and Dimensionless Time

$$\frac{\partial \pi}{\partial t} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{m} a_{i,j} \frac{\partial^2 \pi}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial \pi}{\partial x_i} = 0$$

$$\pi(x, T; y, T) = \delta(x - y)$$

Covariance matrix of state variables

$$a_{i,j} = \sum_{k=1}^{m} \sigma_{i,k} \sigma_{j,k}$$

$$(\bar{\sigma}^2) = E\left\{\frac{1}{n} \sum_{i=1}^{n} a_{i,j}\right\}$$

$$\tau = (\bar{\sigma}^2) t$$

volatility of S&P=0.15
t=1 yr. corresponds to\(\tau=0.0225<<1\)
Varadhan Asymptotics for the Diffusion Kernel

\[
\lim_{\tau \to 0} \tau \ln \pi(x,0;y,T) = -\frac{L^\tau(x,y)}{2} \quad \tau = \sqrt[\sigma]{T},
\]

\[
L(x,y) = \text{geodesic distance between } x \text{ and } y
\]

\[
\pi(x,0;y,T) \approx c(\tau) e^{-(L(x,y))^2/2\tau} \quad \tau \ll 1
\]

\[
(dL)^2 = \sum_{j=1}^{n} g_{\tau,j}(x)dx_j dx_j
\]

Heuristically: Diffusion Kernels "resemble" Gaussian Kernels with \(|x-y|\) replaced by \(L(x,y)\)

We shall use this approximation to compute option prices and implied volatilities assuming tau is small.
Example 1: Local volatility model

\[ \frac{dF}{F_t} = \sigma(F_t, t) dW_t \]

\[ x = \ln \left( \frac{F}{F_0} \right) \]

\[ dx_i = \sigma(x_i, t) dW_i + (...) dt \]

\[ (dL)^2 = \left( \frac{\sigma^2}{\sigma(x, 0)} \right)^2 \frac{dx^2}{\sigma(x)} = \left( \frac{dx}{\sigma(x)} \right)^2 \]

\[ L(x, y) = \left| \int_{x}^{y} \frac{du}{\sigma(u)} \right| = |G(y) - G(x)| \]

1-dimensional distances are always 'trivial'

Special solvable 2-D case: the CEV Model

\[ \sigma(F, t) = \bar{\sigma} \left( \frac{F}{F_0} \right)^{\beta} \]

\[ \sigma(x, t) = \bar{\sigma} e^{\beta \ln x} \]

\[ L(x, y) = \left| e^{-\beta x} - e^{-\beta y} \right| \]

Negative beta for Equities (leverage)

Distance= area under the curve
Stochastic Volatility Models

\[
\frac{dF_t}{F_t} = \sigma_t dW_t
\]

Forward price

\[
\frac{d\sigma_t}{\sigma_t} = \kappa \frac{\sigma_t - \sigma_0}{\sigma_t} dt + \sigma_0 dZ_t
\]

Stochastic vol.

\[
E\{dW_t dZ_t\} = \rho dt
\]

Leverage

\[
\beta = \frac{\kappa \rho}{\sigma}
\]

Beta= regression coefficient of vol on stock returns

Equivalent Model with Independent Brownian Motions (SABR)

\[
\sigma_t = \sigma_t^{(0)} \exp(\beta x_t)
\]

``CEV'' with stochastic independent volatility is equivalent to SV model with correlated volatility, from the Riemann viewpoint

``Parametric leverage''

SV for tails
Riemann Metric for SV / SABR: The Poincare Upper Half-Space Model

The Poincare Upper Half-Space Model

\[ \eta \equiv \kappa \left( 1 - e^{-\beta \eta} \right), \quad \sigma \equiv \sigma^0 \]

\[ dL^2 = \frac{\sigma^2}{\kappa^2} \left( d\eta^2 + \frac{d\sigma^2}{\sigma^2} \right) \]

Geodesics are half-circles with center on the horizontal axis

\[ L(P, Q) = \frac{\sigma}{\kappa} \int_{\frac{\sigma}{\kappa}}^\eta \frac{d\theta}{\sin \theta} \]

Using the asymptotics to compute option prices

\[ F_T = F(x_T), \quad P_0 = F(0) < K \]

\[ \text{CALL} = \int_{K^n} \max(F(y) - K, 0) \pi(0,0; y, T) d^n y \]

\[ \approx c \int_{K^n} \max(F(y) - K, 0) e^{\frac{L^2(0,y)}{2\tau}} d^n y \]

\[ \approx c \int \left\{ \begin{array}{l}
\int_{F(y) < K} \left( F(y) - K \right) e^{\frac{L^2(0,y)}{2\tau}} d^n y \\
\int_{F(y) > K} \left[ \ln \left( \frac{1}{F(y) - K} \right) + \frac{L^2(0,y)}{2\tau} \right] d^n y
\end{array} \right\} \]

\[ \approx c \int e^{\frac{L^2(0,y)}{2\tau}} d^n y \]
Steepest-descent approximation for computing implied volatilities

\[
\int e^{-\int \frac{1}{2} \frac{L^2(y,0)}{\tau} \, dy} \approx e^{-\int \frac{1}{2} \frac{L^2(y,0)}{2\tau} \, dy} \quad \text{for} \quad \tau \ll 1
\]

Equate formulas for OTM calls with Black-Scholes ...

\[
L^*(K) = \min \left\{ L(0,y) \mid y : F(y) > K \right\}
\]

Minimum distance from 0 to the region \( \{ F(y) > 0 \} \)

Small-tau asymptotics (model)

Small-tau asymptotics (Black-Scholes)
Approximation for Implied Volatility for general diffusion model

\[ \sigma_{\text{imp}}(K) = \frac{\ln(K/F_0)}{\min\{L(0, y) \mid y : F(y) > K\}} \]

Example 1: Local Volatility Model

\[ L_{\gamma}(x, y) \equiv \min_{y=0} \int_{y=0}^{1} \sum_{q=1}^{n} \left( a_{ij} - \gamma(t) \right) \gamma(t) dt \]

Implied Volatility = Harmonic Mean of Local Volatility

Berestycki, Busca and Florent, 2001
Example 2: Constant Elasticity of Variance

\[ \sigma(x,t) = \sigma_0 e^{\beta x} \]

Implied volatility

\[ \text{CEV} \]

Example 3: Stochastic Volatility / SABR

\[ \sigma_{\text{imp}}(x) = \frac{\text{ln} \left( \frac{K}{F} \right)}{\text{Implied volatility}} \]

\[ x = \ln \left( \frac{K}{F} \right) \]
Minimizing the distance to the line eta = const. in the Poincare plane

Example 4: the Heston Model
A variant of the Poincare Half-Space

Note: V, not V squared
Closed-form solution for geodesics

\[ \xi = \kappa \left( 1 - e^{-\frac{\beta \theta}{\kappa}} \right) \]

\[ dL^2 = \frac{d\xi^2 + dV^2}{\kappa^2 V} \]

\[ \xi(\theta) = \frac{R^2}{2} (\theta - \sin \theta \cos \theta) + \xi(0) \]

\[ V(\theta) = R^2 \sin^2 \theta \]

\[ 0 \leq \theta \leq \pi \]

Geodesics are cycloids

\[ dL = \frac{2R^2}{\kappa} \sin \theta d\theta \]

Implied volatility curve for Heston model is obtained as an algebraic system

\[ \xi = \frac{\sigma^2_0}{\sin^2 \theta_{\text{init}}} \left( \frac{\pi}{2} - \theta_{\text{init}} + \sin \theta_{\text{init}} \cos \theta_{\text{init}} \right) \]

\[ \sigma(\xi) = \frac{\kappa |\xi| \sin^2 \theta_{\text{init}}}{2\sigma^2_0 \cos \theta_{\text{init}}} \]

Given \( x_i \), solve for \( \theta_{\text{init}} \), and substitute in the second equation
Auto-calibration of SABR and Heston

\[ \sigma_0 = 20\% \quad \kappa_{\text{sabr}} = 0.5 \]

\[ \beta = -4 \quad \kappa_{\text{Heston}} = 2\sigma_0 \kappa_{\text{sabr}} = 0.2 \]

Multi-Asset Derivatives
Multi-Asset Derivatives: Index Options, Rainbows

Derive index volatility skew from single-stock skews and correlation matrix

\[ dx_i = \sigma(x_i, t) dW_i, \quad i = 1, 2, \ldots, n \]

\[ E(dW_i dW_j) = \rho_{ij} dt \]

\[ I = \sum_{i=1}^{\infty} W_i S_i = \sum_{i=1}^{\infty} w_i S_i(0) e^{x_i} \]

\[ \bar{x} = \ln \left( \frac{F_I}{I(0)} \right) \]

N equations for the index components

BBH: ETF of 20 Biotechnology Stocks (Components of IBH)

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Implied Volatility Skews
Multiple Names, Multiple Expirations

AMGN
BGEN
MEDI

BBH March 2003 Implied Vols
Pricing Date: Jan 22 03 10:42 AM

What is the `fair value' of the index volatility reconstructed from the components?
Riemannian metric for the multi-D local vol model

If correlations are constant, the metric is "flat": it is Euclidean metric after making the change of variables $x \rightarrow y$.

Geodesics are straight lines in the $y$-coordinates.

If $\rho_{ij}$ is constant, the metric is Euclidean after the change of variables $x \rightarrow y$.

Steepest Descent = Most Likely Stock Price Configuration

Replace conditional distribution by "Dirac function" at most likely configuration.
Exact solution: Euler-Lagrange Equations

\[ \sigma_{\text{approx}, i}(x) = \frac{\sigma}{\sqrt{\sum_{i=1}^{n} (q_i(x))^2}} \]

\[ \frac{\mathrm{d}}{\mathrm{d}u} \left[ \sum_{i=1}^{n} (q_i(x))^2 \right] \frac{\mathrm{d}u}{\sigma(u,0)} = \sum_{i=1}^{n} \rho_0 \rho_{ij}(x_i) \sigma_i(x_{ji},0) \quad i = 1, \ldots, n \]

Euler - Lagrange equations

Approximate solution: introduce the stock betas

\[ x_i = \beta_i x + \varepsilon_i \]

\[ x_i^* = \beta_i^* x \]

Regression relation between stock and index returns

\[ \sigma_{\text{approx}, i}(x) \approx \sqrt{\sum_{j=1}^{n} \rho_{ij}(x_i)^2 \beta_{j}(x_i)^2} \]

Approximate formula for the optimal stock configuration

\[ \sigma_{\text{approx}, i}(x) \approx \sqrt{\sum_{i=1}^{n} \rho_{ij} p_i p_j \sigma_{\text{approx}, i}(x_i) \sigma_{\text{approx}, j}(x_j)} \]

Performs well in the range -0.2 < x < +0.2
DJX: Dow Jones Industrial Average

T=1 month

DJX Nov 02  Pricing Date: 10/25/02

T=2 months

DJX Dec 02  Pricing Date: 10/25/02
**T=7 months**

**DJX June 03  Pricing Date: 10/25/02**

![Graph showing volatility and volume for DJX June 03 options]

**BBH: Biotechnology HLDR**

**T = 1 month**

**BBH Nov 02 Date: Oct 25 02**

![Graph showing volatility and volume for BBH Nov 02 options]
T = 2 months

BBH Dec 02 Date: Oct 25 02

T = 6 months

BBH Apr 03 Date: Oct 25 02

Is dimensionless time is too long? (Error bars: Juyoung Lim)
Is correlation causing the discrepancy?
S&P 100 Index Options
(Quote date: Aug 20, 2002)

Expiration: Sep 02

Expiration: Oct 02
Implied Correlation: a single correlation coefficient consistent with index vol

\[
\left( \sigma_i^{\text{impl}} \right)^2 = \sum_{i=1}^{N} p_i^2 \sigma_i^{2\text{impl}} + \rho \sum_{i<j}^{N} p_i p_j \sigma_i \sigma_j^{\text{impl}}
\]

\[
\therefore \quad \bar{\rho} = \frac{\left( \sigma_i^{\text{impl}} \right)^2 - \sum_{i=1}^{N} p_i^2 \sigma_i^{2\text{impl}}}{\sum_{i\neq j}^{N} p_i p_j \sigma_i \sigma_j^{\text{impl}}} = \frac{\left( \sigma_i^{\text{impl}} \right)^2 - \sum_{i=1}^{N} p_i^2 \sigma_i^{2\text{impl}}}{\left( \sum_{i=1}^{N} p_i \sigma_i^{\text{impl}} \right)^2} - \sum_{i=1}^{N} p_i^2 \sigma_i^{2\text{impl}}
\]

Approximate formula:

\[
\bar{\rho} = \left( \frac{\sigma_i^{\text{impl}}}{\sum_{i=1}^{N} p_i \sigma_i^{\text{impl}}} \right)^2
\]

Implied correlation can be defined for different strikes, using SDA

Dow Jones Index

Historical Correlation | 3 month Implied Correlation | Index Price
Dow Jones Index: Correlation Skew

Quote Date 9/1/1998 Spot price=78.26

Quote Date 12/10/2001 Spot=99.21
A model for "Correlation skew": Stochastic Volatility Systems

\[
\frac{dS_i}{S_i} = \sigma_i dW_i \\
\frac{d\sigma_i}{\sigma_i} = \kappa_i dZ_i \\
E(dW_i dW_j) = \rho_{ij} dt \\
E(dW_i dZ_j) = r_{ij} dt
\]

\[-\frac{dI}{I}, \quad x_i = \frac{dS_i}{S_i}, \quad y_i = \frac{d\sigma_i}{\sigma_i} \]

Look for most likely configuration of stocks and vols \((x_1, \ldots, x_n, y_1, \ldots, y_n)\) corresponding to a given index displacement \(x\)
Most likely configuration for Stochastic Volatility Systems

\[ x_i^* = \beta_i x \quad \beta_i = \frac{\sigma_i \rho_{il}}{\sigma_I} \]

\[ y_i^* = \gamma_i x \quad \gamma_i = \frac{\kappa_i r_{il}}{\sigma_I} \]

\[ \sigma_{I,\text{loc}}^2(x,t) \equiv \sum_{ij=1}^{n} p_i p_j \sigma_i(0,t) \sigma_j(0,t) e^{\gamma_i \tilde{x}} e^{\gamma_j \tilde{x}} \rho_{ij} \]

Method I: Dupire & Most Likely Configuration for Stock Moves

- **Step 1**: Local volatility for each stock consistent with options market
- **Step 2**: Find most likely configuration for stocks
Method II: Stochastic Volatility System and joint MLC for Stocks and Volatilities

N-dimensional Equity market

\[ \sigma_{I,\text{loc}}(\bar{x},t) \]

- Only one step: compute the most likely configuration of stocks and volatilities at the same time

Methods I and II are not `equivalent'

- Dupire local vol. for single names
  \[ \sigma_{I,\text{loc}}(x_{i,t}) = \sigma_{i}(0,t)e^{\sigma_{i}x_{t}} \]
  \[ \varpi_{i} = \frac{\kappa_{i}\varrho_{i}}{\sigma_{i}} \]

- Index vol., Method I
  \[ \sigma_{I,\text{loc}}^{2}(\bar{x},t) = \sum_{ij}p_{i}p_{j}\sigma_{i}(0,t)\sigma_{j}(0,t)\rho_{ij}e^{\varpi_{i}\bar{x}_{i}^{\sigma_{i}}\varpi_{j}\bar{x}_{j}^{\rho_{ij}}} \]

- Index vol., Method II
  \[ \sigma_{I,\text{loc}}^{2}(\bar{x},t) = \sum_{ij}p_{i}p_{j}\sigma_{i}(0,t)\sigma_{j}(0,t)\rho_{ij}e^{\varpi_{i}\bar{x}_{i}^{\gamma_{i}}\varpi_{j}\bar{x}_{j}^{\gamma_{ij}}} \]
Stochastic Volatility Systems give rise to Index-dependent correlations

\[ \sigma^2_{I,\text{loc}}(x,t) = \sum_{ij} p_i p_j \sigma_i(0,t) \sigma_j(0,t) \rho_{ij} e^{\gamma_i x} e^{\gamma_j x} \]

Method II

\[ = \sum_{ij} p_i p_j \sigma_i(0,t) e^{\beta_i x} \sigma_j(0,t) e^{\beta_j x} \rho_{ij} e^{\gamma_i x} e^{\gamma_j x} e^{-\beta_i x} e^{-\beta_j x} \]

\[ = \sum_{ij} p_i p_j \sigma_{I,\text{loc}}(\beta_i x, t) \sigma_{I,\text{loc}}(\beta_j x, t) \rho_{ij}(x) \]

\[ \rho_{ij}(x) = \rho_{ij} e^{(\gamma_i + \gamma_j - \beta_i \sigma_i - \beta_j \sigma_j) x} \]

Equivalence holds only under additional assumptions on stock-volatility correlations

\[ \sigma_i \beta_i = \frac{\kappa_i r_{ii}}{\sigma_i} \sigma_i \rho_{ii} = \frac{\kappa_i r_{ii}}{\sigma_i} \rho_{ii} \]

Method I

\[ \gamma_i = \frac{\kappa_i r_{ii}}{\sigma_i} \]

Method II

\[ r_{ii} = r_{ii} \rho_{ii} \]

\[ r_{ij} = r_{ii} \rho_{ij} \]

Conditions under which both methods give equivalent valuations
Open (and very doable) problems

- Apply this technology for pricing swaptions based on the volatility skew of LIBOR rates or forward rates.
- If we use a Local Volatility model (e.g., BGM with square-root volatility), the answer is identical to the previous formula.
- The "full" SABR multi-asset model gives rise to a complicated Riemannian metric.
- Credit default models for pricing CDOs are amenable to the same approach, especially copula-type models. I am not aware of any solutions.

Epilogue: Structural Credit Model

\[ x = (x_1, \ldots, x_n) \] vector of firm values

Firm \( i \) defaults before time \( T \) if \( x_i(T) < \alpha \).

Equal weighted CDO: loss of \( m \) dollars if

\[ x(T) \in \Omega_m = \bigcup_{\text{card}(I) \leq m} \cap \{x : x_i < \alpha\} \]

Solve

\[ \inf \{L(0, x) : x \in \Omega_m\} \]