1. A penta–diagonal matrix $A$ is a square matrix for which all elements $a_{ij} = 0$ for all $|i - j| > 2$.

   (a) Show that a linear system $Ax = b$ with $n$ equations and unknowns and an invertible penta–diagonal $A$ can be solved in order $n$ operations and using only a fixed number of vectors of length $n$ for storage. Discuss both the case when no pivoting is needed and the case when pivoting is required.

   (b) Suppose that $A$ is a symmetric matrix. When can we successfully use Cholesky’s method?

   (c) Assume that all elements of $A$ are less than or equal to 1 in absolute value and that $A$ is symmetric. What can then be said about the elements of the triangular factors that result from a successful use of Cholesky’s method?

2. Consider Householder matrices of the form $Q = I - (2/v^Tv)vv^T$, where $v$ is a column vector with $n$ real components and $I$ is the $n \times n$ identity matrix.

   (a) Show that $Q^2 = I$.

   (b) Consider the matrix

   $$
   A = \begin{pmatrix}
   1 & 1 & 1 & 1 \\
   1 & 2 & 1 & 1 \\
   1 & 1 & 3 & 1 \\
   1 & 1 & 1 & 4
   \end{pmatrix}.
   $$

   Find a vector $v$ so that $QA$ has zero entries in the 21, 31, and 41 positions.

   (c) Find a vector $v$ such that $QAQ$ has zero entries in the 31 and 41 positions.

3. Consider vectors $x$ and $y$, with $n$ real components, and matrices $A$ and $B$ which are $n$–by–$n$.

   (a) What is the $\infty$–norm of $x$ and how do we define the $\infty$–norm $\|A\|_\infty$ of $A$ subordinate to the vector $\infty$–norm?
(b) Show that $\|x + y\|_\infty \leq \|x\|_\infty + \|y\|_\infty$.

(c) Show that $\|Ax\|_\infty \leq \|A\|_\infty \|x\|_\infty$.

(d) Show that $\|AB\|_\infty \leq \|A\|_\infty \|B\|_\infty$.

(e) Show that $\|A + B\|_\infty \leq \|A\|_\infty + \|B\|_\infty$.

4. Consider the standard Gaussian elimination algorithm for solving $Ax = b$, where $A$ is a real, square matrix.

(a) What is partial pivoting?

(b) What happens if one uses partial pivoting for a matrix with elements $a_{ij}$ if all elements with $i + j < n$ vanish?

(c) Does the algorithm reveal if the matrix is singular? Discuss the general case and the special matrices just introduced.

(d) Gaussian elimination is a recursive algorithm. Explain.

5. (a) Solve $x^2 = 2$ by Newton’s method. Start from $x_0 = 1$ and take two steps.

(b) Draw a picture related to this specific problem and use it to prove that we have convergence from any initial value.

(c) What is known generally about the convergence of Newton’s method? In particular, do we always have convergence? What can be said about the rate of convergence, in particular, for the simple problem given above.