1. Consider the midpoint rule of numerical integration

\[(b - a)f((a + b)/2)\]

which provides an approximation of \(\int_a^b f(x)dx\).

Show that it gives the exact answer for any linear function but that there are quadratic polynomials for which fails to be exact.

Any linear function can be written as \(A(x - (a + b)/2) + B\). Its integral equals \((b - a)B\), which is also the result of using the midpoint rule.

When applied to \((x - (a + b)/2)^2\), the midpoint rule gives us 0, while the exact answer is clearly a positive number.

2. The first three terms of the Newton interpolation formula are

\[f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)\]

This is a polynomial of degree two.

(a) What is the result if we use this formula with \(x_0 = 0\), \(x_1 = 1\), \(x_2 = -1\) and \(f(x) = x^2 - 1\)?

(b) What is the next term in the formula, resulting in a cubic polynomial, for a general \(f(x)\), with an additional interpolation point \(x_3\)?

(c) What is the result if we use this new formula with \(x_0 = 0\), \(x_1 = 1\), \(x_2 = -1\), \(x_3 = 2\), again with \(f(x) = x^2 - 1\)?

(a) Since the function \(f(x)\) is a polynomial of degree 2, the interpolation formula will just reproduce \(f(x)\). This follows from the uniqueness of the polynomial interpolation problem.

(b) The next term is \(f[x_0, x_1, x_2, x_3](x - x_0)(x - x_1)(x - x_2)\).

(c) Again the degree 3 interpolant of this quadratic polynomial will just recover the given polynomial.